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INFINITESIMAL VARIATIONS PRESERVING THE RICCI TENSOR OF GENERIC SUBMANIFOLDS OF AN ODD-DIMENSIONAL SPHERE

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0. Introduction

Recently many authors have studied the so-called generic submanifold of an odd-dimensional unit sphere $S^{2m+1}(1)$ under the condition that the induced structure on the submanifold is normal ([2]) or antinormal ([4], [6]).

On the other hand, K. Yano, J.S. Pak and one of the present authors have studied infinitesimal variations of a Riemannian manifold ([7], [9]) and those of hypersurfaces of a Sasakian manifold ([8]), and proved the following Theorem A([2]) and B([9]).

THEOREM A ([2]). *Let M be an n-dimensional complete generic submanifold with flat normal connection of an odd-dimensional unit sphere $S^{2m+1}(1)$ and let the Sasakian structure vector defined on $S^{2m+1}(1)$ be tangent to M. If the structure induced on M is normal and the mean curvature vector of M is parallel in the normal bundle, then M is a pythagorean product of the form*

$$S^{p_1}(r_1)\times\cdots\cdots\times S^{p_N}(r_N),$$

where $p_1, \cdots, p_N$ are odd number $\geqq 1$, $r_1^2+\cdots+r_N^2=1$, $N=2m+2-n$, $(N\neq n+2)$, $S^p(r)$ being p-dimensional sphere with radius $r>0$.

THEOREM B ([9]). *Let M^n be a complete hypersurface with constant mean curvature of a unit sphere. If an infinitesimal normal and parallel variation $\bar{x}^h=x^h+\mu C^{h\varepsilon}$, $\mu>0$, preserves the Ricci tensor of M^n, then M^n is a sphere S^n or $S^p\times S^{n-p}$.*

The main purpose of the present paper is to characterize generic submanifolds M of an odd-dimensional unit sphere $S^{2m+1}(1)$ with infinitesimal normal and parallel variation which preserves the Ricci tensor of M.

1. Preliminaries

Let $S^{2m+1}(1)$ be a $(2m+1)$-dimensional unit sphere covered by a system of coordinate neighborhoods $\{U: y^h\}$ and $(F_j^{\ h}, G_{ji}, V^h)$ the set of structure tensors of $S^{2m+1}(1)$, that is, $F_j^{\ h}$ being the Sasakian structure tensor of type (1,1), G_{ji} the Riemannian metric tensor of $S^{2m+1}(1)$ and V^h the Sasakian structure vector, where here and in the sequal, the indices h, i, j and k run over the range $\{1, 2, 3, \cdots, (2n+1)\}$.

Let M be an n-dimensional Riemannian manifold covered by a system of coordinate neighborhoods $\{V: x^a\}$ and isometrically immersed in $S^{2m+1}(1)$ by the immersion $i: M \longrightarrow S^{2m+1}(1)$. We identify $i(M)$ with M itself and represent the immersion locally by $y^h = y^h(x^a)$, where here and throughout this paper the indices a, b, c, d and e run over the range $\{1, 2, 3, \cdots, n\}$. If we put $B_a^{\ h} = \partial_a y^h$, $\partial_a = \partial/\partial x^a$, then $B_a^{\ h}$ are n-linearly independent vectors of $S^{2m+1}(1)$ tangent to M. Denoting by g_{cb} the fundamental metric tensor of M, we have

$$g_{cb} = G_{ji} B_c^{\ j} B_b^{\ i}, \tag{1.1}$$

because the immersion is isometric. We represent by $N_x^{\ h}$ $p(=2m+1-n)$ mutually orthogonal unit normals to M. Then we have $G_{ji} B_b^{\ j} N_x^{\ i} = 0$ and $G_{ji} N_x^{\ j} N_y^{\ i} = g_{xy}$, g_{xy} being the fundamental metric tensor of the normal bundle. In what follows we denote by p the codimension of M and the indices x, y, z, u, v and w run over the range $\{1^*, 2^*, \cdots, p^*\}$.

A submanifold M of $S^{2m+1}(1)$ is called a *generic* (an *anti-holomorphic*) submanifold if the normal space $N_p(M)$ of M at any point $p \in M$ is mapped into the tangent space $T_p(M)$ by action of the structure tensor F of $S^{2m+1}(1)$, that is, $FN_p(M) \subset T_p(M)$ for each point $p \in M$ ([2], [4], [5], [6]).

In this case, we can put in each coordinate neighborhood

$$F_i^{\ h} B_b^{\ i} = f_b^{\ a} B_a^{\ h} - f_b^{\ x} N_x^{\ h}, \quad F_i^{\ h} N_x^{\ i} = f_x^{\ a} B_a^{\ h}, \tag{1.2}$$

where f_b^a is a tensor field of type (1,1) defined on M, f_c^x a local 1-form for each fixed index x and $f_x^a = f_c^y g^{ca} g_{xy}$. Also, we can put the Sasakian structure vector V^h of the form

$$V^h = f^a B_a^{\ h} + f^x N_x^{\ h}, \tag{1.3}$$

f^a and f^x being vector fields defined on M and normal bundle of M respectively.

Now applying the operator F to (1.2) and (1.3) and using the definition of the Sasakian structure tensors, we easily verify that ([2], [4], [5], [6])

$$(1.4)\qquad \begin{cases} f_c^{\ e} f_e^{\ a} = -\delta_c^a + f_c^{\ x} f_x^{\ a} + f_c f^a, \ f_c^{\ e} f_e^{\ x} = -f_c f^x, \\ f^e f_e^{\ a} = -f^x f_x^{\ a}, \ f_x^{\ e} f_e^{\ y} = \delta_x^y - f_x f^y, \ f_e f^e + f_x f^x = 1, \\ f^e f_e^{\ y} = 0, \ g_{de} f_c^{\ d} f_b^{\ e} = g_{cb} - f_c^{\ x} f_{xb} - f_c f_b, \end{cases}$$

where $f_c = f^e g_{ce}$ and $f_x = f^y g_{yx}$.

Denoting $f_{cb} = f_c^{\ a} g_{ba}$, $f_{cx} = f_c^{\ y} g_{yx}$ and $f_{xc} = f_x^{\ a} g_{ac}$, then we can see from (1.4) that $f_{cb} = -f_{bc}$ and $f_{cx} = f_{xc}$.

Denoting by ∇_c the operator of van der Waerden-Borotolotti covariant differentiation with respect to the Christoffel symbols formed with g_{cb}, it is well known that ([2], [4], [5], [6])

$$(1.5)\qquad \nabla_c f_b^{\ a} = -g_{cb} f^a + \delta_c^{\ a} f_b + h_{c\ b}^{\ x} f_x^{\ a} - h_{c\ x}^{\ a} f_b^{\ x}$$

$$(1.6)\qquad \nabla_c f_b^{\ x} = g_{cb} f^x + h_{ce}^{\ \ x} f_b^{\ e},$$

$$(1.7)\qquad \nabla_c f_b = f_{cb} + h_{cb}^{\ \ x} f_x,$$

$$(1.8)\qquad \nabla_c f^x = -f_c^{\ x} - h_{ce}^{\ \ x} f^e,$$

$$(1.9)\qquad h_{cex} f^{ey} = h_{ce}^{\ \ y} f_x^{\ e},$$

where $h_{cb}^{\ \ x}$ is the second fundamental tensor of M and $h_{c\ x}^{\ a} = h_{cb}^{\ \ y} g_{yx} g^{ba}$, $(g^{ba}) = (g_{ba})^{-1}$.

The aggrate $(f_c^a,\ g_{cb},\ f_c^x,\ f^a,\ f^x)$ satisfying (1.4) is said to be *normal*(*partially integrable*) if

$$(1.10)\qquad h_{ce}^{\ \ x} f_b^{\ e} + h_{eb}^{\ \ x} f_c^{\ e} = 0,$$

$$(1.11)\qquad f_c^{\ e} \nabla_e f_b^{\ x} - f_b^{\ e} \nabla_e f_c^{\ x} - (\nabla_c f_b^{\ x} - \nabla_b f_c^{\ x}) f_e^{\ x} - (\nabla_c f_b - \nabla_b f_c) f^x = 0$$

holds respectively ([2], [3]).

Since $S^{2m+1}(1)$ is unit sphere, equations of Gauss, Codazzi and Ricci are respectively

$$(1.12)\qquad K_{dcb}^{\ \ \ \ a} = \delta_d^a g_{cb} - \delta_c^a g_{db} + h_{d\ x}^{\ a} h_{c\ b}^{\ x} - h_{cx}^{\ \ a} h_{d\ b}^{\ x},$$

$$(1.13)\qquad \nabla_d h_{c\ b}^{\ x} - \nabla_c h_{d\ b}^{\ x} = 0,$$

$$(1.14)\qquad K_{dcy}^{\ \ \ \ x} = h_{d\ e}^{\ x} h_{c\ y}^{\ e} - h_{c\ e}^{\ x} h_{d\ y}^{\ e},$$

$K_{dcb}{}^{a}$ and $K_{dcy}{}^{x}$ being the curvature tensor M and that of the normal connection of M respectively.

2. Infinitesimal normal and parallel variations of generic submanifolds of an odd-dimensional sphere

We now consider an infinitesimal variation of the submanifold M of $S^{2m+1}(1)$ given by

$$\bar{y}^{h}=y^{h}+\xi^{h}(x)\varepsilon, \tag{2.1}$$

ξ^{h} being a vector of $S^{2m+1}(1)$ defined along M and ε is an infinitesimal. We now put in each coordinate neighborhood

$$\xi^{h}=\xi^{a}B_{a}{}^{h}+\xi^{x}N_{x}{}^{h}, \tag{2.2}$$

where ξ^{a} is a vector field on M and ξ^{x} a function for each fixed index x.

When $\xi^{a}=0$, that is, when the variation vector ξ^{h} is normal to the submanifold we say that the variation is *normal*, and when the tangent space at a point (y^{h}) of the submanifold and that at the correspoinding point $(\bar{y}^{h})$ of the submanifold are parallel, we say that the variation is *parallel* ([7], [9]).

In order for a normal variation of a submanifold to be parallel, it is necessary and sufficient that

$$\nabla_{c}\xi^{x}=0, \tag{2.3}$$

that is, the variation vector $\xi^{x}N_{x}{}^{h}$ is parallel in the normal bundle ([7]).

In this case, we have from the Ricci identity for ξ^{x}

$$0=\nabla_{d}\nabla_{c}\xi^{x}-\nabla_{c}\nabla_{d}\xi^{x}=K_{dcy}{}^{x}\xi^{y}.$$

Thus if the submanifold M admits p linearly independent infinitesimal normal and parallel variations, then we have $K_{dcy}{}^{x}=0$.

LEMMA 1 ([9]). *Let M be an n-dimensional submanifold of an odd-dimensional unit sphere $S^{2m+1}(1)$. If the submanifold M admits $2m+1-n$ linearly independent infinitesimal normal and parallel variation preserving the Ricci tensor of M, then we have*

$$h_{y}h_{c}{}^{y}{}_{e}h_{b}{}^{e}{}_{x}+nh_{cbx}-h_{dey}h^{de}{}_{x}h_{c}{}^{y}{}_{b}-h_{x}g_{cb}=0, \tag{2.4}$$

where $h_{y}=g^{cb}h_{cby}$ being the mean curvature vector of M.

From now on we assume that the induced structure satisfying (1.4) on M is partially intergrable. Then we have

$$(2.5)\qquad (h_{cey}f^{ex})f_b^{\ y}=(h_{bey}f^{ex})f_c^{\ y}+f_c^{\ x}f_b-f_b^{\ x}f_c.$$

Transvecting (2.5) with $f_z^{\ b}$ and using (1.4), we find

$$(2.6)\qquad h_{cez}f^{ex}-(h_{cey}f^y)f^{ex}f_z=P_{zy}^{\ \ x}f_c^{\ y}-\delta_z^{\ x}f_c+f_zf^xf_c,$$

where we have put

$$(2.7)\qquad P_{zy}^{\ \ x}=h_{bey}f^{ex}f_z^{\ b},$$

from which, transvecting f^z and denoting $\rho^2=f_xf^x$,

$$(1-\rho^2)(h_{cey}f^y)f^{ex}=P_{zy}^{\ \ x}f^zf_c^{\ y}-(1-\rho^2)f^xf_c.$$

Substituting this into (2.6), we find

$$(2.8)\qquad (1-\rho^2)h_{cez}f^{ex}=-(1-\rho^2)\delta_z^{\ x}f_c+\{(1-\rho^2)P_{zy}^{\ \ x}+f_zP_{yu}^{\ \ x}f^w\}f_c^{\ y}.$$

Putting $P_{zyx}=P_{zy}^{\ \ w}g_{wx}$, then P_{zyx} is symmetric for any index because of (1.9) and (2.7). If we take the skew-symmetric part with respect to x and z and use (1.9), then we obtain from (2.8)

$$(2.9)\qquad (f_zP_{ywx}f^w-f_xP_{ywz}f^w)f_c^{\ y}=0.$$

If we assume that the function $1-\rho^2$ does not vanish almost everywhere on M, then (2.8) gives

$$(2.10)\qquad h_{e\ x}^{\ c}f_y^{\ e}=R_{yz}^{\ \ x}f_c^{\ z}-\delta_y^{\ x}f_c,$$

where we have put

$$(2.11)\qquad R_{yzx}=P_{yzx}+1/1-\rho^2f_zP_{ywx}f^w.$$

Transvecting (2.9) with $f_u^{\ c}$ and $f_a^{\ c}$ respectively and combining these equations, we find

$$(2.22)\qquad (f_zP_{ywx}-f_xP_{ywz})f^w=0.$$

This means that R_{xyz} is symmetric for any index.

If the normal connection of M is flat, that is, $K_{dcy}^{\ \ \ x}=0$, by transvecting (1.14) with $f_z^{\ c}$ and making use of (2.10), we find

$$(2.13)\qquad (R_{wz}^{\ \ x}R_{vy}^{\ \ w}-R_{wyz}R_v^{\ xw})f_d^{\ v}=\delta_z^{\ x}(h_{dey}f^e)-g_{yz}(h_{de}^{\ \ x}f^e).$$

First of all, we prove

LEMMA 2. *Let M be a generic submanifold with flat normal connection of an odd-dimensional unit sphere* $S^{2m+1}(1)$. *If the induced structure on M is partially integrable and the function* $1-f_x f^x$ *does not vanish almost everywhere, then we have* $f^x=0$ *or* $p=1$.

PROOF. Transvecting (2.13) with $f^u f_u^d$ and using (1.4) and (2.10), we get

$$(2.14)\qquad (R_{wz}{}^x R_{vy}{}^w - R_{wyz}R_v{}^{xw})f^v = g_{yz}f^x - \delta_z^x f^y$$

because the function $1-\rho^2$ is nonzero almost everywhere.

If we transvect (2.13) with f_x and use (2.10) and (2.14), then we get

$$(2.15)\qquad g_{yz}f^x(h_{dex}f^e + f_{dx}) = f_z(h_{dey}f^e + f_{dy}),$$

from which, contract with respect to y and z

$$(2.16)\qquad (p-1)(h_{dex}f^e + f_{dx})f^x = 0.$$

If we take the skew-symmetric part with respect to y and z of (2.15), then we have

$$f_z(h_{dey}f^e + f_{dy}) = f_y(h_{edz}f^e + f_{dz}),$$

which, transvect with f^z and use (2.16),

$$(2.17)\qquad \rho^2(h_{dey}f^e + f_{dy})(p-1) = 0.$$

Transvecting (2.17) with f^{dy} and using (1.4) and (2.10), we have $\rho^4(p-1)^2 = 0$. Therefore, Lemma 2 is proved.

LEMMA 3. *Let M be an* $n(\neq 2m)$*-dimensional generic submanifold of an odd-dimensional unit sphere* $S^{2m+1}(1)$. *Suppose that M admits* $2m+1-n$ *linearly independent infinitesimal normal and parallel variations preserving the Ricci tensor of M and the induced structure on M is partially integrable. If the function* $1-f_x f^x$ *does not vanish almost everywhere, then the induced structure on M is normal.*

PROOF. Since $p\neq 1$, we see from Lemma 2 that f^x vanishes identically on M. We have the identity

$$(2.18)\qquad \nabla^b[f_x{}^c\nabla_c f_b{}^x] = \frac{1}{2}\|\nabla_c f_b{}^x + \nabla_b f_c{}^x\|^2 - \|\nabla_c f_b{}^x\|^2 + f_x{}^c\nabla^b\nabla_c f_b{}^x.$$

Transvecting (1.6) with f_x^c and using (1.4) and (1.8) with $f^x=0$, we find $f_x^c\nabla_c f_b^x=0$.

Now, computing the length of square of $\nabla_c f_b^x$, we have

$$\|\nabla_c f_b^{\ x}\|^2 = h_{c\ b}^{\ x} h^{cb}_{\ \ x} - h_{ce}^{\ \ x} h_{a\ x}^{\ c} f_y^{\ e} f_a^{\ y} - p. \tag{2.19}$$

On the other hand, from the Ricci identity, we have

$$\nabla_d \nabla_c f_b^{\ x} - \nabla_c \nabla_d f_b^{\ x} = -K_{dcb}^{\ \ \ e} f_e^{\ x},$$

which implies

$$(\nabla^b \nabla_c f_b^{\ x}) f_x^{\ c} = K_{cb} f^{bx} f_x^{\ c}$$

because of (1.6) with $f^x = 0$. Thus, it follows that

$$(\nabla^b \nabla_c f_b^{\ x}) f_x^{\ c} = (n-1)p + h_x R^x - h_{c\ e}^{\ x} h_{b\ x}^{\ e} f_y^{\ c} f^{by}, \tag{2.20}$$

where $R_x = R_{yx}^{\ \ \ y}$.

Substituting (2.19) and (2.20) into (2.18), we get

$$\frac{1}{2}\|\nabla_c f_b^{\ x} + \nabla_b f_c^{\ x}\|^2 - h_{cb}^{\ \ x} h^{c\ b}_{\ x} + h_x R^x + np = 0.$$

Transvecting (2.4) with f^b and using (1.8) and (2.10) with $f^x=0$, we get

$$h_{cbx} h^{c\ b}_{\ y} - h_z R_{x\ y}^{\ z} - n g_{xy} = 0.$$

The last two relationships give

$$\nabla_c f_b^{\ x} + \nabla_b f_c^{\ x} = 0.$$

Thus, (1.10) holds because of (1.6) with $f^x=0$. This completes the proof of the lemma.

Combining Theorem A, B and Lemma 2 and 3, we conclude

THEOREM 4. *Let M be an n-dimensional complete generic submanifold of an odd-dimensional unit sphere $S^{2m+1}(1)$. Suppose that M admits $2m+1-n$ linearly independent infinitesimal normal and parallel variation preserving the Ricci tensor of M, the induced structure on M is partially integrable and the function $1-f_x f^x$ does not vanish almost everywhere. If the mean curvature vector of M is parallel in the normal bundle, then M is*

$$S^{2m}(r),\ S^p(r_1)\times S^{2m-p}(r_2)\ or\ S^{p_1}(r_1)\times\cdots\times S^{p_N}(r_N),$$

where $p_1, \cdots, p_N$ are odd number $\geqq 1$, $r_1^2+\cdots+r_N^2=1$, $N=2m+2-n$, $(N\neq n+2)$.

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