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CONVERGENCES OF GAMES BETTER WITH TIME

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1. Introduction

L.H. Blake [1, 2] introduced the concept of games fairer with time and established a fundamental L_1 convergence theorem.

A. Mucci [3, 4] introduced the notion of martingales in the limit and proved an a.e. convergence for L_1 bounded martingales in the limit, these processes are a special case of games fairer with time.

Recently Blake [5] again defined new concept of weak submartingales in the limit and it was proved that a uniformly integrable weak submartingale in the limit has an L_1 limit.

The purpose of this paper is to introduce a notion of games better with time which is a generalization of both games fairer with time and weak submartingales in the limit. We also obtain L_1 convergence theorem and a.e. convergence on atomic set.

2. Convergence theorem

Throughout this paper, let $(\Omega, \mathscr{F}, P)$ be a probability space and $(\mathscr{F}_n)_{n=1}$ an increasing sequence of sub-σ-fields of $\mathscr{F}$ and $(X_n)_{n=1}$ be a sequence of random variables adapted to $(\mathscr{F}_n)_{n=1}$.

A stopping time is a random variable τ assuming positive integer values and the value $+\infty$, such that $\{\tau=n\}\in\mathscr{F}_n$ for each n.

DEFINITION [2]. $(X_n)_{n=1}$ is a *game fairer with time* if for every $\varepsilon>0$ $P(|E[X_n|\mathscr{F}_m]-X_m|>-\varepsilon)\longrightarrow 0$ as $n, m\longrightarrow\infty$ with $n>m$.

DEFINITION [5]. $(X_n)_{n=1}$ is a *weak submartingale in the limit* if, for every $\varepsilon>0$, there exist M such that for $n>m>M$

$$P(E[X_n|\mathscr{F}_m]-X_m\geq 0)\geq 1-\varepsilon.$$

DEFINITION 1. $(X_n)_{n=1}$ is called a *game better with time* if for every $\varepsilon>0$, there exist M such that for any $n>m>M$

$$P(E[X_n|\mathscr{F}_m]-X_m\geq -\varepsilon)\geq 1-\varepsilon.$$

We easily can find an example of a game better with time which is neither a game fairer with time nor a weak submartingale in the limit.

First we will show the L_1 convergence theorem.

THEOREM 2. *If* $(X_n)_{n=1}$ *is a uniformly integrable game better with time, then* $(X_n)_{n=1}$ *has an* L_1 *limit.*

Before the proof is presented, the following two lemmas are necessary.

We omit the proofs of the two lemmas.

LEMMA 3. *If a sequence* $\{a_n\}_{n=1}$ *of real numbers is bounded with the property that for every* $\varepsilon>0$ *there exists a positive integer* M *such that whenever* $p>q>M$ $a_p-a_q>-\varepsilon$, *then the sequence* $\{a_n\}_{n=1}$ *has a limit.*

LEMMA 4. *If* $(X_n)_{n=1}$ *is a uniformly integrable game better with time, then for every* $\varepsilon>0$ *there exists a positive integer* M *such that whenever* $p>q>M$

$$\int_{\left[B_{p,q}\left(\frac{\varepsilon}{2}\right)\right]^c}(|X_p|+|X_q|)<\frac{\varepsilon}{4},$$

where $$B_{p,q}\left(\frac{\varepsilon}{2}\right)\equiv\left\{E[X_p|\mathscr{F}_q]-X_q\geq-\frac{\varepsilon}{2}\right\},$$

and $\{\int_A X_p\}_{p\geq n}$ *is bounded for each* $A\in\mathscr{F}_n$ *for every* n.

PROOF. Define a sequence of signed measures $\{\mu_n\}_{n=1}$ where μ_n is defined on $\mathscr{F}_n$ by

$$\mu_n(A)=\int_A X_n dP, \quad A\in\mathscr{F}_n.$$

For each $A\in\mathscr{F}_n$, $\lim_{\substack{p\to\infty\\ p\geq n}}\mu_p(A)$ exists. Indeed, consider for any $\varepsilon>0$

$$\mu_p(A)-\mu_q(A)\geq\mu_p\left(A\cap\left[B_{p,q}\left(\frac{\varepsilon}{2}\right)\right]^c\right)-\mu_q\left(A\cap\left[B_{p,q}\left(\frac{\varepsilon}{2}\right)\right]^c\right)-\frac{\varepsilon}{2}$$

for all p, q for which $A\in\mathscr{F}_n$ with $p>q>n$. By Lemma 3 & 4, the sequence $\{\mu_p(A)\}$ of reals converges. Hence, let $\nu_n(A)\equiv\lim_{\substack{p\to\infty\\ p\geq n}}\mu_p(A)$ for every $A\in\mathscr{F}_n$.

Since $(X_n)_{n=1}$ is a uniformly integrable sequence

$$|\nu_n(A)|<\infty \text{ for all } A\in\mathscr{F}_n,$$

and so ν_n is a signed measure on $\mathscr{F}_n$ by Vital-Hahn-Saks theorem.

It is clear that $\nu_n\ll P$ for each n, and so there exists a sequence $(Y_n)_{n=1}$

which is a martingale and $\nu_n(A)=\int_A Y_n dP$ for all $A\in\mathscr{F}_n$. It is important to note that $(Y_n)_{n=1}$ is a uniformly integrable sequence. This follows exactly as in [6 : p590]. Hence, $(Y_n)_{n=1}$ converges in the L_1 norm.

The proof will be completed by showing that

$$\int|X_m-Y_m|\longrightarrow 0 \text{ as } n\longrightarrow\infty.$$

To this end, write

$$\int|X_m-Y_m|=\int_{C_m}(X_m-Y_m)+\int_{C_m^c}(Y_m-X_m),$$

and
$$\int X_m-\int Y_m=\int_{C_m}(X_m-Y_m)+\int_{C_m^c}(X_m-Y_m),$$

where
$$C_m\equiv\{X_m-Y_m\geq 0\}.$$

Clearly
$$\int(X_m-Y_m)\longrightarrow 0 \text{ as } m\longrightarrow\infty.$$

Hence the proof will be completed by showing that

$$\int_{C_m}(X_m-Y_m)\longrightarrow 0 \text{ as } m\longrightarrow\infty.$$

$$0\leq\int_{C_m}(X_m-Y_m)=\int_{C_m\cap B_{n,m}\left(\frac{\varepsilon}{2}\right)}(X_m-Y_m)+\int_{C_m\cap\left[B_{n,m}\left(\frac{\varepsilon}{2}\right)\right]^c}(X_m-Y_m)$$
$$\leq\int_{C_m\cap B_{n,m}\left(\frac{\varepsilon}{2}\right)}(X_n-Y_n)+\int_{C_m\cap\left[B_{n,m}\left(\frac{\varepsilon}{2}\right)\right]^c}(X_m-Y_m)+\frac{\varepsilon}{2}$$
$$\leq\int_{C_n}(X_n-Y_n)+\int_{C_m\cap\left[B_{n,m}\left(\frac{\varepsilon}{2}\right)\right]^c}(X_m-Y_m)+\frac{\varepsilon}{2} \quad \text{where } n>m.$$

It follows from Lemma 3 & 4 that $\operatorname{Lim}_{m\to\infty}\int_{C_m}(X_m-Y_m)$ exists.

Thus, we should show that this limit is zero.

Suppose not: that is, there exists some $\gamma>0$ and M_γ such that for all $m>M_\gamma$

$$\int_{C_m}(X_m-Y_m)>\gamma.$$

Consider

$$\int_{C_m}X_m\leq\int_{C_m\cap B_{m+k,m}\left(\frac{\gamma}{4}\right)}X_{m+k}+\int_{C_m\cap\left[B_{m+k,m}\left(\frac{\gamma}{4}\right)\right]^c}X_m+\frac{\gamma}{4}$$
$$\leq\int_{C_m\cap B_{m+k,m}\left(\frac{\gamma}{4}\right)}X_{m+k}-\int_{C_m\cap\left[B_{m+k,m}\left(\frac{\gamma}{4}\right)\right]^c}X_m+\frac{\gamma}{4}+\frac{\gamma}{4}$$
$$\leq\int_{C_m\cap B_{m+k,m}\left(\frac{\gamma}{4}\right)}X_{m+k}-\int_{C_m\cap\left[B_{m+k,m}\left(\frac{\gamma}{4}\right)\right]^c}X_{m+k}+\frac{\gamma}{4}+\frac{\gamma}{4}$$
$$\leq\int_{C_m\cap B_{m+k,m}\left(\frac{\gamma}{4}\right)}X_{m+k}+\int_{C_m\cap\left[B_{m+k,m}\left(\frac{\gamma}{4}\right)\right]^c}X_{m+k}+\frac{\gamma}{4}+\frac{\gamma}{4}+\frac{\gamma}{4}=\int_{C_m}X_{m+k}+\frac{3}{4}\gamma$$

for all $m>\max\,[M_\gamma, N]$, where M in Lemma 5 is replaced by N when we substitute $\frac{\gamma}{2}$ for ε.

Thus for all $k\geq 1$ and $m>\max\ [M_\gamma, N]$

$$\gamma<\int_{C_m}(X_m-Y_m)\leq\cdots\leq\int_{C_m}X_{m+k}-\int_{C_m}Y_m+\frac{3}{4}\gamma.$$

A contradiction arises as $k\to\infty$. So, $\lim_{m\to\infty}\int_{C_m}(X_m-Y_m)=0$ and the theorem is proved.

Secondarily we obtain a.e. convergence theorem for games better with time on atomic set.

DEFINITION [7]. Let F be a family of real-valued measurable functions $f:\Omega\longrightarrow R$ defined on a probability space $(\Omega,\mathscr{F},P)$. Let g be a measurable function such that

a) $g\leq f$ a.e. for all $f\in F$,

b) if h is a measurable function such that $h\leq f$ a.e. for all $f\in F$, then $h\leq g$ a.e.

This function g, which is the greatest lower bound of the family F in the sense of a.e. inequality is denote by ess inf(F).

The following two lemmas are necessary for proving the theorem.

LEMMA 5. *If $(X_n)_{n=1}$ is a game better with time, A is an atom of $\mathscr{F}$, $A_n=$ ess inf $\{B|B\in\mathscr{F}_n, A\subset B\}$ and* $\lim\sup X=a>b=\lim\inf X$ *on A, where $a,b\in R\cup\{\infty,-\infty\}$, then for every $t\in N$ there exist m such that $t<m$ and $A_m\subsetneqq A_t$.*

PROOF. Assume for every $k\geq t$ $A_k=A_{k+1}$, put $P(A)=\alpha$ and $a-b=\beta$, We first prove the lemma in the case of $\beta<\infty$. Let $\varepsilon>0$ such that $0<\varepsilon<\min\left\{\alpha, \frac{1}{2}\beta\right\}$. By definition of a game better with time, there exists M such that $t\leq M$ and $P(E[X_n|\mathscr{F}_m]-X_m>-\varepsilon)\geq 1-\varepsilon$ for every n, m with $n>m>M$. Since $P(A)>\varepsilon$ and $E[X_n|\mathscr{F}_m]-X_n$ is constant on A_t for any $n>m>M>t$ we have $E[X_n|\mathscr{F}_m]-X_m>-\varepsilon$ on A.

On the other hand, there exist $n_1, n_2>M$ such that $n_1<n_2$ and $X_{n_1}-X_{n_2}>\frac{3}{4}\beta$ on A. Then $\varepsilon<\frac{3}{4}\beta<X_{n_1}-X_{n_2}<\varepsilon$ on A. It is contradiction.

In the case of $\beta=\infty$ we can easily prove the lemma.

LEMMA 6. *If $(X_n)_{n=1}$ is a game better with time, A is an atom on $(\Omega,\mathscr{F},P)$ and $A_n=$*ess inf$\{B|B\in\mathscr{F}_n, A\subset B\}$ *then for every $\varepsilon>0$ there exist $M\in N$ such that*

$\inf_{m\geq t} E[X_m|\mathscr{F}_t]-X_t\geq -\varepsilon$ *on* A_t *for every* $t\geq M$.

PROOF. It is sufficient to prove the lemma for sufficiently small $\varepsilon>0$. Take ε such that $P(A)>\varepsilon>0$. Then there exist M such that $P(E[X_n|\mathscr{F}_m]-X_m>-\varepsilon)>1-\varepsilon$ for every m, n such that $n\geq m\geq M$. Since for every $m\geq t\geq M$ $E[X_m|\mathscr{F}_t]-X_t$ is constant on A_t and $P(A_t)>\varepsilon$, $E[X_m|\mathscr{F}_t]-X_t\geq -\varepsilon$ on A_t for all $m\geq t$. Therefore $\inf_{m\geq t} E[X_m|\mathscr{F}_t]-X_t\geq -\varepsilon$ on A_t.

THEOREM 7. *Let* $(X_n)_{n=1}$ *be a game better with time such that* $\int_{(\tau<\infty)}X_\tau^+<\infty$ *for all stopping time* τ *and* A *is an atom of the probability space* $(\Omega.\ \mathscr{F},\ P)$, *then* $\lim_{n\to\infty} X_n$ *exists and* $>-\infty$ *a. e. on* A.

PROOF. Every random variable is constant a.e. on every atom. So we can put $X_n=a_n$ a.e. $(n=1, 2, \cdots)$ on the atom A where $P(A)>0$ and a_n are real constants. Put $A_n=\text{ess inf}\,\{B|B\in\mathscr{F}_n,\ A\subset B\}$. Clearly, $A_n\in\mathscr{F}_n$, $A_n\supset A_{n+1}$, $A_n\supset A$ and A_n is atom of $\mathscr{F}_n$ for all n. Suppose that $\lim X_n$ does not exist on A. Then $\limsup X_n=a>b=\liminf X_n$ on A for some a, b. We first prove the theorem in the case of $a-b$ is finite. Then by above lemmas given $\varepsilon>0$, there exists an integer n_1 such that

$$|a_{n_1}-a|<\frac{\beta}{4},\ \inf_{m\geq n_1} E[X_m|\mathscr{F}_{n_1}]-X_{n_1}>-\frac{\varepsilon}{2}\ \text{on}\ A_{n_1}$$

and we can find an integer n_2 such that

$$|a_{n_2}-b|<\frac{\beta}{4},\ A_{n_2}\subsetneqq A_{n_1}\ \text{and}\ \inf_{m\geq n_2} E[X_m|\mathscr{F}_{n_2}]-X_{n_2}>-\frac{\varepsilon}{2^2}.$$

Continuing this process by induction, we can take integers n_{2k-1}, n_{2k}, $(k=2, 3, \cdots)$ such that

$$n_{2k-1}<n_{2k},\ |a_{n_{2k-1}}-a|<\frac{\beta}{4},\ A_{n_{2k-2}}\supsetneqq A_{n_{2k-1}},$$

$$\inf_{m\geq n_{2k-1}} E[X_m|\mathscr{F}_{n_{2k-1}}]-X_{n_{2k-1}}\geq -\frac{\varepsilon}{2^{2k-1}}\ \text{on}\ A_{n_{2k-1}},$$

$$|a_{n_{2k}}-b|<\frac{\beta}{4},\ A_{n_{2k-1}}\supsetneqq A_{n_{2k}},\ \text{and}\ \inf_{m\geq n_{2k}} E[X_m|\mathscr{F}_{n_{2k}}]-X_{n_{2k}}\geq \frac{\varepsilon}{2^{2k}}\ \text{on}\ A_{n_{2k}}.$$

Then we have

$$\int_{A_{n_{2k-1}}-A_{n_{2k}}}X_{n_{2k}}=\int_{A_{n_{2k-1}}}X_{n_{2k}}-\int_{A_{n_{2k}}}X_{n_{2k}}=\int_{A_{n_{2k-1}}}X_{n_{2k}}-P(A_{n_{2k}})a_{n_{2k}}$$

$$=\int_{A_{n_{2k-1}}} E\,[X_{n_{2k}}|\mathscr{F}_{n_{2k-1}}]-P(A_{n_{2k}})a_{n_{2k}}$$
$$\geq\int_{A_{n_{2k-1}}}\Big(X_{n_{2k-1}}-\frac{\varepsilon}{2^{2k-1}}\Big)-P(A_{n_{2k}})a_{n_{2k}}$$
$$\geq P(A_{n_{2k-1}})a_{n_{2k-1}}-P(A_{n_{2k}})a_{n_{2k}}-\frac{\varepsilon}{2^{2k-1}}$$
$$\geq(a_{n_{2k-1}}-a_{n_{2k}})P(A_{n_{2k}})-\frac{\varepsilon}{2^{2k-1}}$$
$$\geq\frac{1}{2}\,\beta P(A)-\frac{\varepsilon}{2^{2k-1}}\geq\frac{1}{2}\,\beta P(A)-\frac{\varepsilon}{2}$$

Define $\tau=n_i$ on $A_{n_i}-A_{n_{i-1}}$ $(i=2,3,\cdots)$ and $\tau=\infty$ elsewhere. Then τ is a stopping time. Take ε such that $0<\varepsilon<\beta P(A)$.

$$\int_{(\tau<\infty)}X_\tau^+=\sum_{i=2}^{\infty}\int_{A_{n_i}-A_{n_{i-1}}}X_{n_i}^+\geq\sum_{k=1}^{\infty}\int_{A_{n_{2k-1}}-A_{n_{2k}}}X_{n_{2k}}$$
$$\geq\sum_{k=1}^{\infty}\Big\{\frac{1}{2}\beta P(A)-\frac{\varepsilon}{2^{2k-1}}\Big\}\geq\sum_{k=1}^{\infty}\Big(\frac{1}{2}\,\beta P(A)-\frac{\varepsilon}{2}\Big)=\infty.$$

In second case of $a-b=\beta=\infty$, we can take $a_{n_i}(i=1,2,\cdots)$ such that $a_{n_{2k-1}}-a_{n_{2k}}\geq 1$ $(k=1,2,3,\cdots)$ and others are the same to the first case. Then we also can have $\int_{(\tau<\infty)}X_\tau^+=\infty$. This contradicts the assumption and we proved that $\lim_{n\to\infty}X_n$ exists a.e. on A. Now suppose $\lim_{n\to\infty}X_n=-\infty$ on A. Then there exists an integer n_1 such that $a_{n_1}<0$ and $\inf_{m\geq n_1}E[X_m|\mathscr{F}_{n_1}]-X_{n_1}\geq-\frac{\varepsilon}{2}$ on A_{n_1} and we fine an integer n_2 such that $n_1<n_2$

$$A_{n_1}\supsetneqq A_{n_2},\ a_{n_2}<\frac{1}{P(A)}[a_{n_1}P(A_{n_1})-P(A_n)]$$

and $$\inf_{m\geq n_2}E[X_m|\mathscr{F}_{n_2}]-X_{n_2}>-\frac{\varepsilon}{4}$$

on A_{n_2}. By induction we can take a sequence $\{n_k\}$ such that

$$n_{k-1}<n_k,\ A_{n_{k-1}}\supsetneqq A_{n_k},\ a_{n_k}<\frac{1}{P(A)}[a_{n_{k-1}}P(A_{n_{k-1}})-P(A_{n_{k-1}})]$$

and $\inf_{m\geq n_k}E[X_m|\mathscr{F}_{n_k}]-X_{n_k}>-\frac{\varepsilon}{2^k}$ on A_{n_k}. Then we have

$$\int_{A_{n_{k-1}}-A_{n_k}}X_{n_k}=\int_{A_{n_{k-1}}}X_{n_k}-\int_{A_{n_k}}X_{n_k}\geq\int_{A_{n_{k-1}}}E\,[X_{n_k}|\mathscr{F}_{n_{k-1}}]-P(A_{n_k})a_{n_k}$$
$$\geq\int_{A_{n_{k-1}}}\Big(X_{n_{k-1}}-\frac{\varepsilon}{2^{k-1}}\Big)-P(A_{n_k})\,a_{n_k}$$

$$\geq P(A_{n_{k-1}})a_{n_{k-1}} - P(A_{n_k})a_{n_k} - \frac{\varepsilon}{2^{k-1}}$$

$$\geq P(A_{n_{k-1}}) - \frac{\varepsilon}{2^{k-1}} \geq P(A) - \frac{\varepsilon}{2^{k-1}} \geq P(A) - \frac{\varepsilon}{2}.$$

Define $\tau = n_1$ on $A_{n_{i-1}} - A_{n_i}$ and $\tau = \infty$ otherwise. Then τ is a stopping time. Take ε such that $0 < \varepsilon < P(A)$

$$\int_{(\tau<\infty)} X_\tau^+ = \sum_{i=1}^{\infty} \int_{A_{n_i} - A_{n_{i-1}}} X_{n_i}^+ \geq \sum_{i=1}^{\infty}\left(P(A) - \frac{\varepsilon}{2^{i-1}}\right) \geq \sum_{i=1}^{\infty}\left(P(A) - \frac{\varepsilon}{2}\right) = \infty$$

This contradicts our assumption.

COROLLARY 8. *Let $(X_n)_{n=1}$ be a game fairer with time such that $\int_{(\tau<\infty)} |X_\tau| < \infty$ for every stopping time τ and an atom A of probability space. Then $\lim_{n\to\infty} X_n$ exists and is finite a.e. on A.*

PROOF. By the similar method, we can easily prove it.

THEOREM 9. *Let $(X_n)_{n=1}$ be a game better with time such that $\sup_n \int |X_n| < \infty$ and let A be an atom of the probability space. Then $\lim_{n\to\infty} X_n$ exists and is finite on A.*

PROOF. Suppose that $\lim_{n\to\infty} X_n$ does not exist on A. Then $\lim\sup X_n = a > b = \lim\inf X_n$ on A for some $a, b \in R$ and clearly $a - b = \beta < \infty$. So we can take the same n_k as in the first case of previous theorem. Then we have for every k with $n_{2k} \leq m$

$$\int_{A_{n_{2k-1}} - A_{n_{2k}}} X_m = \int_{A_{n_{2k-1}} - A_{n_{2k}}} E[X_m | \mathscr{F}_{n_{2k}}] \geq \int_{A_{n_{2k-1}} - A_{n_{2k}}} \left(X_{n_{2k}} - \frac{\varepsilon}{2^{2k}}\right)$$

$$\geq \int_{A_{n_{2k-1}} - A_{n_{2k}}} X_{n_{2k}} - \frac{\varepsilon}{2^{2k}} \geq \int_{A_{n_{2k-1}}} X_{n_{2k}} - \int_{A_{n_{2k}}} X_{n_{2k}} - \frac{\varepsilon}{2^{2k}}$$

$$= \int_{A_{n_{2k-1}}} E[X_{n_{2k}} | F_{n_{2k-1}}] - P(A_{n_{2k}})a_{n_{2k}} - \frac{\varepsilon}{2^{2k}}$$

$$\geq \int_{A_{n_{2k-1}}} \left(X_{n_{2k-1}} - \frac{\varepsilon}{2^{2k-1}}\right) - P(A_{n_{2k}})a_{n_{2k}} - \frac{\varepsilon}{2^{2k}}$$

$$\geq \frac{1}{2}\beta P(A) - \left(\frac{\varepsilon}{2^{2k-1}} + \frac{\varepsilon}{2^{2k}}\right)$$

$$\geq \frac{1}{2}\beta P(A) - \frac{\varepsilon}{2^{2k-2}} \text{ for } n_{2k} \leq m$$

and

$$\int_\Omega |X_m| \geq \int_{A_{n_1}-A_{n_2}} |X_m| + \cdots + \int_{A_{n_{2k-1}}-A_{n_{2k}}} |X_m|$$
$$\geq \left(\frac{1}{2}\beta P(A)-\varepsilon\right)+\cdots+\left(\frac{1}{2}\beta P(A)-\frac{\varepsilon}{2^{2k-2}}\right)\geq\left(\frac{1}{2}\beta P(A)-\varepsilon\right)\cdot k$$

Take ε such that $\frac{1}{2}\beta P(A)>\varepsilon>0$. Then as $k\to\infty$, $\lim\int_\Omega |X_m|=\infty$. This completes the theorem.

Now we consider the following example. It shows that if A is not an atomic set, the a.e. convergence is not assured. To show this we shall construct a counter example. Let $\Omega=[0,1]$ and P be a Lebesque measure on Ω. Define $X_n(x)=1$ if $x\in[k2^{-\nu}, (k+1)2^{-\nu}]$ and $X_n(x)=0$ otherwise, where $n=k+2^\nu$, $0\leq k<2^\nu$. Put $\mathscr{F}_n=\sigma(X_1, \cdots, X_n)$ and $\mathscr{F}=\sigma(\bigcup_{n=1}^{\infty}\mathscr{F}_n)$. Then $\mathscr{F}$ contains no atomic set and $(X_n)_{n=1}$ is a game better with time which does not converge a.e.

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