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A NOTE ON $\mathfrak{A}$-IDEALS

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1. Introduction

In [6], $\mathfrak{A}$-ideals and translations of an universal algebra $\mathfrak{A}$ are introduced by Tae Ho Choe. He used unary algebraic polynomials of $\mathfrak{A}$ to construct $\mathfrak{A}$-ideal. In this paper, we obtain a necessary and sufficient condition between $\mathfrak{A}$-ideal N and congruence relation θ_N induced by N. Finally we find some properties of topological $\mathfrak{A}$-ideal. The terminologies and the definitions used here mainly follow [8] for universal algebras, [4] for category and [2] for topology. Throughout this paper, all universal algebras will be of finite type, i.e., by a τ-algebra we mean a pair $<A,\ F=(f_i)_{i\in I}>$, where A is a set, F a family of operations: maps f_i of A^{mi} into A, where mi is the arity of the f_i, that is nonnegative integer and I is a finite ordered set $\{1, 2, \dots, t\}$. Then we say that it is of arity type $\tau=(m1, m2, \dots mt)$. Throughout this paper, we assume that F has a distinguished nullary operation o, which we call the zero of $\mathfrak{A}$.

Denote $\mathfrak{A}[x]$ to be the set of all unary algebraic polynomials over $\mathfrak{A}$.

2. $\mathfrak{A}$-ideals

LEMMA 2.1. *Let $\mathfrak{A}$ and $\mathscr{L}$ be τ-algebras and h a homomorphism of $\mathfrak{A}$ into $\mathscr{L}$. Then for any unary algebraic polynomial $p(x)\in\mathfrak{A}[x]$ there exists $q(x)\in\mathscr{L}[x]$ such that $hp(x)=q(x)h$, i.e., $h(p(a))=q(h(a))$ for all $a\in A$.*

LEMMA 2.2. *Let $\mathfrak{A}$ and $\mathscr{L}$ be τ-algebras and h a homomorphism of $\mathfrak{A}$ onto $\mathscr{L}$. Then for any $q(x)\in\mathscr{L}[x]$ there exists $p(x)\in\mathfrak{A}[x]$ such that $hp(x)=q(x)h$.*

THEOREM 2.1. *Let $\mathfrak{A}$ and $\mathscr{L}$ be τ-algebras and h a homomorphism of $\mathfrak{A}$ into $\mathscr{L}$. Let M be a $\mathscr{L}$-ideal. Then $h^{-1}(M)$ is also $\mathfrak{A}$-ideal.*

PROOF. For any $p(x)\in\mathfrak{A}[x]$ if $p(a)\in h^{-1}(M)$ for some $a\in h^{-1}(M)$. By Lemma 2.1, there exists $q(x)\in\mathscr{L}[x]$ such that $hp(x)=q(x)h$. Then $h(p(a))=q(h(a))\in M$. If $b\in h^{-1}(M)$ then $h(p(b))=q(h(b))\in M$.

THEOREM. 2.2. *Let $\mathfrak{A}$ and $\mathscr{L}$ be τ-algebras and h a homomorphism of $\mathfrak{A}$ onto $\mathscr{L}$. Let N be an $\mathfrak{A}$-ideal. If Ker $h \subset \theta_N$ then $h(N)$ is a $\mathscr{L}$-ideal.*

PROOF. For any $q(x) \in \mathscr{L}[x]$ if $q(b) \in h(N)$ for some $b \in h(N)$ then there exists $a \in N$ such that $h(a)=b$. Then by Lemma 2.2 there exists $p(x) \in \mathfrak{A}[x]$ such that $hp(x)=q(x)h$, thus $h(p(a))=q(b) \in h(N)$. Then $(p(a), n) \in Ker\ h$ for some $n \in N$. Since $Ker\ h \subset \theta_N$, $p(a) \in N$. Then for any $d \in h(N) q(d)=q(h(e))=h(p(e)) \in h(N)$, where $e \in N$ and $h(e)=d$.

THEOREM 2.3. *The intersection of any $\mathfrak{A}$-ideals of τ-algebra is an $\mathfrak{A}$-ideal.*

Therefore $\{N_i | i \in I\}$ forms a complete lattice, its greatest element is $\mathfrak{A}$ and least element is $\bigcap_{i \in I} N_i$. Denote $\mathfrak{A}_0$ to be the set of all $\mathfrak{A}$-ideals (about 0).

DEFINITION 2.2. [6] Let $\mathfrak{A}$ be a τ-algebra and let $\phi = \{\phi_a(x) \in \mathfrak{A}[x] | a \in A\}$ be a set of nonidentically zero unary algebraic polynomials such that $\phi_a(x)=0$ has the unique solution a in $\mathfrak{A}$. Such a ϕ is called a *translation* (about 0) of $\mathfrak{A}$.

DEFINITION 2.3. [6] Let $\mathfrak{A}$ be a τ-algebra and let ϕ be a translation of $\mathfrak{A}$. ϕ is said to be *left invertible* if $\phi_a(x) \in \phi$ has a left inverse $\Psi_a(x) \in \mathfrak{A}[x]$ (in the sense that $\Psi_a(\phi_a(x))=x$) and $\phi_0(x)=x$.

3. Topological τ-algebras

DEFINITION 3.1. [7] A topological τ-algebra is an object $\langle A, (f_i)_{i \in I}\ \mathscr{T} \rangle$, where $\langle A, (f_i)_{i \in I} \rangle$ is a τ-algebra and $\mathscr{T}$ is a Hausdorff topology on A such that each f_i is a continuous map of the product space $(A^{mi}, \mathscr{T}^{mi})$ into $(A, \mathscr{T})$.

THEOREM 3.1. *Let $(\mathfrak{A}, \mathscr{T})$ be a topological τ-algebra and let C be a component of O then C is $\mathfrak{A}$-ideal.*

PROOF. For any $p(x) \in \mathfrak{A}[x]$ if $p(a) \in C$ for some $a \in C$, then $p(b) \in C$ for all $b \in C$. Suppose $p(b) \notin C$ for some $b \in C$. Since each operation is continuous $p(x)$: $\mathfrak{A} \longrightarrow \mathfrak{A}$ is a continuous map. Then $p(C)$ is connected. Since $p(a) \in p(C) \cap C$, $C \cup p(C)$ is connected and $O \in C \cup p(C)$ and $p(b) \notin C$ but $p(b) \in C \cup p(C)$. Thus C is a proper subset of $C \cup p(C)$. It is a contradiction to the fact that C is a component of O.

THEOREM 3.2. *Let $(\mathfrak{A}, \mathscr{T})$ be a topological τ-algebra and let N be an $\mathfrak{A}$-ideal. Let $\phi = \{\phi_a(x) \in \mathfrak{A}[x] | a \in A\}$ be a left invertibly translation of $\mathfrak{A}$. Then N is open in $\mathfrak{A}$ if and only if θ_N is open in $\mathfrak{A} \times \mathfrak{A}$.*

PROOF. For any $(a, b)\in\theta_N$ and $\phi_a(x)\in\phi$ since $\phi_a(b)\in N$, $\phi_a(a)=0\in N$ and $\phi_a(x):\mathfrak{A}\longrightarrow\mathfrak{A}$ is continuous there exists neighborhood U and V of a and b, respectively, such that $\phi_a(U)\subset N$ and $\phi_a(V)\subset N$. Then $U\times V$ is a neighborhood of (a,b) such that $U\times V\subset\theta_N$. Because for any $(c,d)\in U\times V$ let $p(x)\in\mathfrak{A}[x]$ and $p(c)\in N$. Then there exists $q(x)\in\mathfrak{A}[x]$ such that $p(x)=q(\phi_a(x))$. Since $p(c)=q(\phi_a(a))\in N$ $p(d)=q(\phi_a(d))\in N$. Hence $(c, d)\in\theta_N$. Conversely, if θ_N is open in $\mathfrak{A}$ then $[a]_{\theta_N}$ is open in $\mathfrak{A}$ for each $a\in\mathfrak{A}$. Since $N=[n]\theta_N$ for some $n\in N$, N is open in $\mathfrak{A}$.

THEOREM 3.3. *Let $(\mathfrak{A}, \mathscr{T})$ be a topological τ-algebra and let $\phi=\{\phi_a(x)\in\mathfrak{A}[x]\mid a\in A\}$ be a left invertible translation. Then every open $\mathfrak{A}$-ideal is closed in $\mathfrak{A}$.*

PROOF. Let N be an open $\mathfrak{A}$-ideal. Then by Theorem 3.2 θ_N is open in $\mathfrak{A}\times\mathfrak{A}$. Thus for each $a\in A$ $[a]_{\theta_N}$ is open. Then $N=A-\bigcup_{a\notin N}[a]_{\theta_N}$. Since $\bigcup_{a\notin N}[a]_{\theta_N}$ is open, N is closed in $\mathfrak{A}$.

THEOREM 3.4. *Let $(\mathfrak{A}, \mathscr{T})$ be a topological τ-algebra and let N be an $\mathfrak{A}$-ideal. Let $\phi=\{\phi_a(x)\in\mathfrak{A}[x]\mid a\in A\}$ be a left invertible translation. Then $\mathfrak{A}/N$ is discrete if and only if N is open in $\mathfrak{A}$.*

THEOREM 3.5. *Let $(\mathfrak{A}, \mathscr{T})$ be a topological τ-algebra and let C be a component of O. Let ϕ be a left invertible translation. Then $\dot{O}$ has a component consisted $\dot{O}$ only in $\mathfrak{A}/C$.*

PROOF. Let $\dot{U}$ be a component of $\dot{O}$ in $\mathfrak{A}/C$ and let $\varphi:\mathfrak{A}\longrightarrow\mathfrak{A}/C$ be a canonical map. Suppose there exists $\dot{x}\in\dot{U}$ such that $\dot{x}\neq\dot{O}$. Then $\varphi^{-1}(\dot{U})$ is a subset of $\mathscr{U}$ and $C\subset\varphi^{-1}(\dot{U})$, since for any $c\in C$ $\varphi(c)\in\varphi(C)\subset\dot{U}$. Moreover, C is a proper subset of $\varphi^{-1}(\dot{U})$, since $x\notin C$ but $x\in\varphi^{-1}(\dot{U})$. Since C is a component of O $\varphi^{-1}(\dot{U})$is not connected. Then there exist open subsets P, Q in $\mathfrak{A}$ such that

(1) $\varphi^{-1}(\dot{U})=(P\cap\varphi^{-1}(\dot{U}))\cup(Q\cap\varphi^{-1}(\dot{U}))$, where $(P\cap\varphi^{-1}(\dot{U}))\cap(Q\cap\varphi^{-1}(\dot{U}))=\phi$ and neither set is empty. Then it is verified that $\dot{U}=(\varphi(P)\cap\dot{U})\cup(\varphi(Q)\cap\dot{U})$. For any $a\in U$ since $\varphi^{-1}(\dot{U})=\bigcup_{a\in U}[a]_{\theta_c}[a]_{\theta_c}\subset\bigcup_{a\in U}[a]_{\theta_c}$. Then by (1) $[a]_{\theta_c}=(P\cap[a]_{\theta_c})\cup(Q\cap[a]_{\theta_c})$. Since $\mathfrak{A}$ has a left invertible translation $\Psi_a([O]_{\theta_c})=\Psi_a(C)=[a]_{\theta_c}$, since $b\in[a]_{\theta_c}\Leftrightarrow(b,a)\in\theta_c\Leftrightarrow\phi_a(b)\in C\Leftrightarrow\Psi_a(\phi_a(b))=b\in\Psi_a(C)$. Since $\Psi_a(x)$ is continuous $[a]_{\theta_c}$ is connected. Then $[a]_{\theta_c}\subset(P\cap[a]_{\theta_c})$ or $[a]_{\theta_c}\subset(Q\cap[a]_{\theta_c})$.

Thus $\varphi(P\cap\varphi^{-1}(\dot{U}))\cap\varphi(Q\cap\varphi^{-1}(\dot{U}))=\phi$. For, suppose there exists $\dot{a}\in\varphi(P\cap\varphi^{-1}(\dot{U}))\cap\varphi(Q\cap\varphi^{-1}(\dot{U}))$. Then $\varphi^{-1}(\dot{a})\subset P$, $\varphi^{-1}(\dot{a})\subset Q$ and $\varphi^{-1}(\dot{a})=[u]\theta c$. Thus $[u]\theta_c\subset P\cap\varphi^{-1}(\dot{U})$ and $[u]\theta_c\subset Q\cap\varphi^{-1}(\dot{U})$. It is a contradiction. Then $(\varphi(P)\cap\varphi(U)=\dot{U})\cap(\varphi(Q)\cap\dot{U})=\phi$ and since φ is an open map, each is open in $\dot{U}$. It is contradictory to the fact that $\dot{U}$ is connected.

THEOREM 3.6. *Let $(\mathfrak{U}, \mathcal{T})$ be a topological τ-algebra with a left invertible translation ϕ. Then an $\mathfrak{U}$-ideal N is closed in $\mathfrak{U}$ if and only if $[a]_{\theta_N}$ is closed in $\mathfrak{U}$, for each $a\in A$.*

PROOF. For any $b\in[a]^C_{\theta_N}$ since $(b, a)\notin\theta_N$ by Proposition 2.5, $\phi_a(b)\notin N$. Since N^C is open in $\mathfrak{U}$ and $\phi_a(x)$ is continuous there exists an open set U in $\mathfrak{U}$ such that $b\in U$ and $\phi_a(U)\subset N^C$. Moreover, $U\subset[a]^C_{\theta_N}$, since for any $u\in U\phi_a(u)\in N^C$ $\phi_a(a)=O\in N$.

THEOREM 3.7. *Let $(\mathfrak{U}, \mathcal{T})$ be a compact τ-algebra with a left invertible translation ϕ. If N be an open $\mathfrak{U}$-ideal then $\varphi:\mathfrak{U}\longrightarrow\mathfrak{U}/N$ is a closed map.*

PROOF. Let H be closed in $\mathfrak{U}$ and let $\dot{a}\in\mathfrak{U}/N-\varphi(H)$, where $\dot{a}=[a]_{\theta_N}$ and $a\notin\bigcup_{h\in H}[h]_{\theta_N}$. Since N is open by Theorem 3.2 θ_N is open in $\mathfrak{U}\times\mathfrak{U}$. Then $\{[h]_{\theta_N}\mid h\in H\}$ is an open covering of H. Since H is compact, there exist $[h1]_{\theta_N},\dots,[hn]_{\theta_N}$ such that $\bigcup_{i=1}^{n}[hi]_{\theta_N}\supset H$. Since N is open and $\mathfrak{U}$ has a left invertible translation ϕ by Theorem 3.3 and 3.6 $[hi]_{\theta_N}$ is closed in $\mathfrak{U}$. Then $\bigcup_{i=1}^{n}[hi]_{\theta_N}$ is closed. Thus there exists an open set U in $\mathscr{U}$ such that $a\in U\subset\mathfrak{U}-\bigcup_{i=1}^{n}[hi]_{\theta_N}$. Since φ is an open map $\varphi(U)$ is a neighborhood of $\dot{a}$. And $\varphi(U)\subset\mathfrak{U}/N-\varphi(\bigcup_{i=1}^{n}[hi]_{\theta_N})\subset\mathfrak{U}/N-\varphi(H)$.

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