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RELATIVE IDEALS IN GROUPS

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Let S be a semigroup and T be a sub-semigroup of S. Now a nonempty subset A of S is called a *left T-ideal* if $TA \subseteq A$ [4]. The right T-ideal is defined analogously. A nonempty subset A of S is called a *T-ideal* if it is both left T-ideal and right T-ideal.

In [4] A.D. Wallace has shown how Faucett's theorem on cut-points of the minimal ideal of a compact connected semigroup may be relativized. Also in [5], he has studied the relativized Green's relation.

Now the object of this paper is to study the relative ideals in groups. The examples given below show that a group may contain relative ideals. With the help of this notion of relative ideals we have obtained a number of criteria for a subsemigroup T of a group S to be a subgroup and also to be a normal subgroup. Also the results obtained in this paper generalise some results on *"Generalised semi ideals of semigroups"* introduced by M.K. Sen [2].

EXAMPLE 1. Let M_2 be the set of all 2×2 nonsingular metrices over the field of rational numbers. Then M_2 is a group w.r.t. matrix multiplication. Let $T=\left\{\begin{pmatrix} a & 0 \\ 0 & b\end{pmatrix}, \text{ where } a, b \text{ are integers}\right\}$ and $A=\left\{\begin{pmatrix} e & f \\ g & h\end{pmatrix}, e, f, g, h, \text{ are even integers}\right\}$ Then A is a left T-ideal as well as a right T-ideal of M_2

EXAMPLE 2. Let R be the multiplicative group of the set of all nonzero rational numbers. Let $T=\{r^2/r\in R\}$. Then T is a subsemigroup of R. Let $A=\left\{\frac{1}{3}r^2/r\in R\right\}$. Then A is a T-ideal of R.

PROPOSITION 1. *Let G be a group and A be a nonempty subset of G. Then there exists a subsemigroup T of G s.t. A is left (right) T-ideal of G.*

PROOF. Let $T=\{t\in G/tA\subseteq A\}$. Obviously, T is nonempty since 1, the identity element of G, belongs to T. Let $t_1, t_2\in T$ then $t_1A\subseteq A$ and $t_2A\subseteq A$. Now $(t_1\cdot t_2)A=t_1(t_2A)\subseteq t_1A\subseteq A$. So $t_1\cdot t_2\in T$. So T is a subsemigroup of G and A is a left T-ideal. Similarly we can show that A is a right T_1-ideal of G for a subsemigroup T_1 of G.

EXAMPLE 3. Let J be the additive group of all integers and J_2 be the set of all positive even integers. Then J_2 is a subsemigroup of J. Let $A=\{a\in J \mid a\geqslant 6\}$. Then A is a J_2-ideal but the complement of A in J is not a J_2-ideal.

PROPOSITION 2. *Let G be a semigroup. A subsemigroup T of G will be a group if complement of every T-ideal (both left and right) is also a T-ideal. Conversely, if complement of every T-ideal is a T-ideal then both T and G are groups.*

PROOF. First we assume that T is a group. Let A be a left T-ideal of G and $x\in G/A$ (the complement of A in G), we shall show that $tx\in G/A$ where $t\in T$. If possible, let $tx\in A$. Then $t^{-1}(tx)\in A$ i.e $x\in A$, which is a contradiction. So $tx\in G/A$ and hence G/A is a left T-ideal. Conversely, we assume that complement of every T-ideal is a T-ideal. Let A be an ideal of the semigroup T. Then A is a T-ideal. Hence T/A will also be a T-ideal. Let $t\in T$ and $a\in A$. Then $ta\in A$. Also $ta\in T/A$ since A and T/A are ideals of T. So T does not contain any proper ideal, (both left and right). So T is a group. Similarly, we can show that G is a group.

COROLLARY 1. *A semigroup S will be a group iff complement of every ideal is an ideal. Following M.K. Sen* [2], *a subset A of a semigroup S will be called a generalised left semiideal (g.l.s.) if* $x^r A\subseteq A$, $\forall x\in S$.

COROLLARY 2. *A commutative semigroup S will be a commutative group iff complement of every g.s. ideal is a g.s. ideal.*

PROPOSITION 3. *Let G be a semigroup with the subsemigroup T. Then T will be a group if the difference A-B of two T-ideals (both left and right) is a T-ideal (assuming that* ϕ, *the empty set is also a T-ideal). Conversely, if the difference of two T-ideals is T-ideal then both T and G will be groups.*

Let $I(S)$ be the set of all T-ideals (both left and right) of a semigroup S and $P(S)$ be the set of all T-ideals A s.t. $ta\in A\Longrightarrow a\in A$ and $at\in A\Longrightarrow a\in A$.

PROPOSITION 4. *A subsemigroup T of a semigroup G will be a group iff* $I(G)=P(G)$.

PROOF. Let the subsemigroup T of G be a group. Obviously, $P(G)\subseteq I(G)$. Let A be a left T-ideal of G and $ta\in A$. Then $a=t^{-1}(ta)\in A$. So $A\in P(G)$. Similarly, if A is a right T-ideal of G. Then $at\in A\Longrightarrow a\in A$. So $A\in P(G)$. So $I(G)=P(G)$. Conversely, let $I(G)=P(G)$. Let A be a left T-ideal of G. We shall show that G/A is also a left T-ideal. Let $a\in G/A$ and $t\in T$. Then $ta\in G/A$. For if $ta\in A$ then $a\in A$ which is a contradiction. So G/A is also a T-ideal. Now the proposition follows from Proposition 2.

PROPOSITION 5. *Let S be a semigroup with* 1 *and T be a subsemigroup with* 1. *Let $P_1(S)$ be the set of all T-ideals of S which contain* 1. *Then $P_1(S)$ is a semigroup with the identity element and zero.*

PROOF. Let A, $B \in P_1(S)$. Then $AB \in P_1(S)$. Since AB is also a T-ideal containing 1. Also $(AB)C = A(BC)$, where $A, B, C \in P_1(S)$. Now $TA \subseteq A$ and also $A \subseteq TA$ since $1 \in T$. Therefore $TA = A$. Similarly $A = AT$. So T is the identity element of $P_1(S)$. Similarly, we can show that $AS = S = SA$. So S is the zero element of $P_1(S)$.

PROPOSITION 6. *Let T be a subgroup of a semigroup S. Then $I(S)$ is a Boolean algebra w.r.t.* $\cup$, $\cap$ *and complementation (we assume that $\phi \in I(S)$).*

PROOF. Proposition follows from Proposition 2.

PROPOSITION 7. *$P(S)$ is a Boolean ring on assuming that $\phi \in P(S)$.*

PROOF. Let $A, B \in P(S)$. Then $A - B \in P(S)$. For if $a \in A - B$ and $t \in T$ then $ta \in A - B$, since $ta \in B \Rightarrow a \in B$ which contradicts our assumption $a \in A - B$. So $A - B \in P(S)$. Also we can show that $A \cup B$, $A \cap B \in P(S)$. So $P(S)$ is a Boolean ring.

PROPOSITION 8. *Let T be a subgroup of a group G. Let P be the collection of all left T-ideal $\{Ta \mid a \in G\}$. Then P is a partition of G.*

PROOF. Obviously, any element x of G belongs to Tx i.e., to some member of P. Also any two members of P are disjoint. On the contrary, if possible, let $Ta \cap Tb \neq \phi$. Let $x \in Ta \cap Tb$. Then there exist elements g_1, g_2 in T s.t. $x = g_1 a = g_2 b$. So $a = g_1^{-1}(g_2\, b) = (g_1^{-1}\, g_2)\, b \in Tb$. Thus $Ta \subseteq Tb$.

Similarly, $Tb \subseteq Ta$. So $Ta = Tb$. Hence P forms a partition of G.

A semigroup S will be called a *left(right) T-simple semigroup* iff S is the only left (right) T-ideal of S. A semigroup which is both left. T-simple and right T-simple is called *T-simple.*

PROPOSITION 9. *A semigroup S will be left T-simple iff for every a in S we have $Ta = S$.*

PROOF. Let us suppose that S is left T-simple and $a \in S$. Then $ta \in Ta$, where $t \in T$. Then for any $t_1 \in T$, $t_1(ta) \in Ta$. So Ta is a left T-ideal of S. Hence $Ta = S$

since S is left T-simple. Conversely, let $Ta=S$ for every a in S. Let A be a left T-ideal of S and $a\in A\subseteq S$. Then $S=Ta\subseteq A\subseteq S$. So $S=A$. Thus S contains no proper left T-ideal. Hence the proposition.

COROLLARY. *A commutative semigroup S will be generalised simple* [3] *iff for every a in S we have* $\bar{S}a=S$ *where* $\bar{S}=\{x^r \mid x\in S\}$.

PROPOSITION 10. *A semigroup S will be a group if it is T-simple.*

PROOF. Proposition follows from the fact that the existence of an ideal of S implies the existence of a T-ideal of S.

Suppose A is a T-ideal of commutative semigroup S. Let $\beta(A)$ denote the set of all those elements a of S for each of which there exists an element $t\in T$ s.t. $ta\in A$. It is clear that $A\subseteq\beta(A)$.

PROPOSITION 11. $\beta(A)$ *is a T-ideal of S. If T is a group then* $\beta(A)=A$. *Conversely, if* $\beta(A)=A$ *for any T-ideal A of S then both T and S will be groups.*

PROOF. Let $a\in\beta(A)$. Then $ta\in A$ for some $t\in T$. Let $t_1\in T$. Then $t_1(ta)\in A$ i.e. $t(t_1\ a)\in A$ so $t_1\ a\in\beta(A)$. Hence $\beta(A)$ is a T-ideal of S. Next, let T be a subgroup of S. Obviously, $A\subseteq\beta(A)$. Let $a\in\beta(A)$. So $ta\in A$ for some $t\in T$. Then $a=t^{-1}(ta)\in A$. So $\beta(A)\subseteq A$. Hence $A=\beta(A)$. Conversely, let $\beta(A)=A$ for every T-ideal A of S. Now Ta is a T-ideal of S where $a\in T$. So $\beta(Ta)=Ta$. But from definition $\beta(Ta)=T$. So $Ta=T$. Hence T is a group. Similarly, we can show that S is a group.

PROPOSITION 12. *A subgroup A of a group G will be left T-ideal iff it is a right T-ideal.*

A semigroup S is said to have the properties α, β or γ if the relation $L_1\cap L_2=L_1L_2$, $R_1\cap R_2=R_1R_2$ or $L_1\cap R_1=L_1R_1$ hold for left T-ideals L_1, L_2 and right T-ideals R_1, R_2 of S.

PROPOSITION 13. *In a semigroup S having property* α (β *or* γ) *every left (right, one sided) T-ideal is a right (left, two sided) T-ideal.*

PROOF. Let S be a semigroup having property α. Let L_1 be a left T-ideal of S. Now $L_1=L_1\cap S=L_1S$ i.e. L_1 is also a right ideal and hence a right T-ideal of S.

PROPOSITION 14. *Let S be a semigroup having property* γ (α *or* β) *and T be a subsemigroup of S. Then T is a normal subsemigroup of S.*

PROOF. Let S be a semigroup having the property $\gamma(\alpha$ or $\beta)$ and $a \in S$. Now Ta is a left T-ideal of S and hence also a right T-ideal of S. Now $Ta = Ta \cap Ta = Ta\ Ta$. Similarly we have $aT = aT \cap aT = aT\ aT$. Since aT is a two sided T-ideal of S, so $T(aT) \subseteq aT$. Hence $Ta\ Ta \subseteq a\ Ta$. So aTa is also a left T-ideal and hence a two sided T-ideal of S. So by property γ. $Ta\ Ta = aTa \cap Ta\ Ta = a\ Ta\ Ta\ Ta$. Similarly $aT\ aT = a\ Ta\ Ta\ Ta$. So $Ta\ Ta = aT\ aT$. Hence $aT = Ta$. So T is normal subsemigroup of S.

COROLLARY. *A subsemigroup T of a group G will be a normal subgroup if the complement of any T-ideal is a T-ideal.*

PROOF. Corollary follows from Proposition 2 and 14.

PROPOSITION 15. *A semigroup S having the property γ is regular.*

PROPOSITION 16. *A semigroup S having the property γ is a semilattice of groups.*

Proposition 15 and 16 follows from Theorems 3 & 4 [1] since every ideal is a T-ideal.

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