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ANALYTICAL STUDIES FOR EVEN SUB-HARMONIC SYNCHRONIZATION OF A WEAKLY NON-LINEAR CONSERVATIVE PHYSICAL SYSTEM

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1. Introduction

Through other non-linear phenomena there exist the phenomenon of synchronization (entrainment of frequency), from the point of view of mathematics the study of this phenomenon reduces to study of harmonic and sub-harmonic solution of differential equation. There exist different approach for treatment of this subject, among these is the analytical approach; the advantage of this approach permited us to obtain an algebraic from of solution without necessity of introducing numerical values for parameters or initial conditions.

After proving the existence of those synchronization [3], and established the natural frequency, the approach amplitude and the approach explicit formula of harmonic and sub-harmonic synchronization of add order [4] of our physical system to $O(\lambda^3)$, this paper is devoted to calculate the approach total phase, the approach amplitude and the approach explicit formula of sub-harmonic synchronization of order $m, (m=2, 4)$ of the previous physical system to $O(\lambda^3)$, these results are compared with the results which are obtained by using the index method which give us a best justification (Fig. d). One of the important analytical method for solving a weakly non-linear differential equation is the generalized synchronization method which makes the subject of the following section.

2. The generalized synchronization method

Consider the following system of n differential equations

$$\frac{dx}{dt}=\lambda f(x, t) \tag{2}$$

where x is a vector in the Euclidean space of dimension n, $f(x, t)$ is a periodic function in t with period T, and an analytical function in x, and λ is a very small parameter, to this system we associated the following reduced system:

$$\frac{dy}{dt}=\lambda \overline{F}(y) \tag{3}$$

where $$x=y+\tilde{G}(y,t) \tag{4}$$
and $-$ and $\sim$ (tilda) means that there exist two functions of different natures, and the functions $\overline{F}(y)$, $\tilde{G}(y,t)$ are defined in the following form:

$$\left.\begin{aligned}\overline{F}(y)&=\overline{F}_1(y)+\lambda\overline{F}_2(y)+\cdots+\lambda^{m-1}\overline{F}_m(y)+\cdots\\ \tilde{G}(y,t)&=\lambda\tilde{G}_1(y,t)+\lambda_2\tilde{G}_2(y,t)+\cdots+\lambda^m\tilde{G}_m(y,t)+\cdots\end{aligned}\right\} \tag{5}$$

where
$$\left.\begin{aligned}\overline{F}_1(y)&=\overline{f(y,t)}\\ \overline{F}_2(y)&=\overline{\frac{\partial \tilde{f}}{\partial y}\int \tilde{f}(y,t)dt}\\ &\vdots\end{aligned}\right\} \tag{6}$$

and
$$\left.\begin{aligned}\tilde{G}_1(y,t)&=\int \tilde{f}(y,t)dt\\ \tilde{G}_2(y,t)&=\int\left\{\frac{\partial f}{\partial y}\tilde{f}\,dt\right\}^{\sim}+\frac{\partial \tilde{f}}{\partial y}\int\left\{(\int\tilde{f}dt)\,dt-\left\{\int\left(\int\frac{\partial\tilde{f}}{\partial y}dt\right)dt\right\}\tilde{f}\right\}\\ &\vdots\end{aligned}\right\} \tag{7}$$

3. Equation in the standard form

When $\lambda=0$ in the previous differential equation, its solution is
$$\left.\begin{aligned}x&=a\cos(\omega t+\phi),\\ \dot{x}&=-a\omega\sin(\omega t+\phi)\end{aligned}\right\} \tag{8}$$
where a, ϕ are constant of integration, when λ is not zero we consider a and ϕ as unknowns functions of time in general solution (8), from (1) and (8), we can deduce the following exact system:
$$\frac{d}{dt}\begin{bmatrix}a(t)\\ \phi(t)\end{bmatrix}=\frac{\lambda a^3(t)}{32\omega}\begin{bmatrix}a(t)(2\sin\phi_1+2\sin\phi_3+3\sin\phi_4+\sin\phi_5+\sin\phi_6\\ 10\cos\phi_1+10\cos\phi_2+5\cos\phi_3+5\cos\phi_4+\cos\phi_5+\cos\phi_6\end{bmatrix} \tag{9}$$
where
$$\left.\begin{aligned}&\phi_1=(\omega+1)t+\phi(t),\ \phi_2=(\omega-1)t+\phi(t),\ \phi_3=(3\omega+1)t+3\phi(t)\\ &\phi_4=(3\omega-1)t+3\phi(t),\ \phi_5=(5\omega+1)t+5\phi(t),\ \phi_6=(5\omega-1)t+5\phi(t)\end{aligned}\right\} \tag{10}$$

when ω is not in the neighbourhood of 1, $\frac{1}{3}$ and $\frac{1}{5}$, one find
$$\begin{aligned}\bar{f}(y,t)=O,\ \tilde{f}(y,t)&=\begin{bmatrix}\tilde{f}_1(y,t)\\ \tilde{f}_2(y,t)\end{bmatrix}\\ &=\frac{A^3}{32\omega}\begin{bmatrix}A(2\sin\phi_1+2\sin\phi_2+3\sin\phi_3+3\sin\phi_4+\sin\phi_5+\sin\phi_6\\ 10\cos\phi_1+10\cos\phi_2+5\cos\phi_3+5\cos\phi_4+\cos\phi_5+\cos\phi_6\end{bmatrix}\end{aligned} \tag{11}$$

Now we calculate the function $F_2(y,t)$, from (6) and (11), we have

$$F_2(y,t)=\begin{bmatrix} \dfrac{\partial \tilde{f}_1}{\partial A}\int \tilde{f}_1(y,\ t)dt+\dfrac{\partial \tilde{f}_1}{\partial \varphi}\int \tilde{f}_2(y,t)dt \\ \dfrac{\partial \tilde{f}_2}{\partial A}\int \tilde{f}_1(y,\ t)dt+\dfrac{\partial \tilde{f}_2}{\partial \phi}\int \tilde{f}_2(y,\ t)dt \end{bmatrix}=\begin{bmatrix} F_{2a}(y,\ t) \\ F_{2\phi}(y,\ t) \end{bmatrix},\ y=\begin{bmatrix} A \\ \phi \end{bmatrix} \tag{12}$$

where

$$\begin{aligned} F_{2a}(y,\ t)=&\frac{A^7}{(32\omega)^2}\Big\{\frac{1}{\omega+1}[2\sin(2\phi_1)+2\sin(\phi_1+\phi_2)-18\sin(\phi_2-\phi_1) \\ &+33\sin(\phi_1+\phi_3)-57\sin(\phi_3-\phi_1)+33\sin(\phi_1+\phi_4)-57\sin(\phi_4-\phi_1) \\ &-29\sin(\phi_5-\phi_1)+21\sin(\phi_1+\phi_5)+21\sin(\phi_1+\phi_6)-29\sin(\phi_6-\phi_1)] \\ &+\frac{1}{\omega-1}2\sin(\phi_1+\phi_2)+18\sin(\phi_2-\phi_1)+2\sin(2\phi_2)+33\sin(\phi_2+\phi_3) \\ &-57\sin(\phi_3-\phi_2)+33\sin(\phi_2+\phi_4)-57\sin(\phi_4-\phi_2)+21\sin(\phi_2+\phi_5) \\ &-29\sin(\phi_5-\phi_2)+21\sin(\phi_2+\phi_6)-29\sin(\phi_6-\phi_2)]+\frac{1}{3\omega+1}\Big[-7\sin(\phi_1+\phi_3) \\ &+17\sin(\phi_3-\phi_1)-7\sin(\phi_2+\phi_3)+17\sin(\phi_3-\phi_2)+\frac{9}{2}\sin(2\phi_3) \\ &+\frac{9}{2}\sin(\phi_3+\phi_4)-\frac{81}{2}\sin(\phi_4-\phi_3)+\frac{13}{2}\sin(\phi_3+\phi_5)-\frac{37}{2}\sin(\phi_5-\phi_3) \\ &+\frac{13}{2}\sin(\phi_3+\phi_6)-\frac{37}{2}\sin(\phi_6-\phi_3)\Big]+\frac{1}{3\omega-1}\Big[-7\sin(\phi_1+\phi_4) \\ &+17\sin(\phi_4-\phi_1)-7\sin(\phi_2+\phi_4)+17\sin(\phi_4-\phi_2)+\frac{9}{2}\sin(\phi_3+\phi_4) \\ &+\frac{81}{2}\sin(\phi_4-\phi_3)+\frac{9}{2}\sin(2\phi_4)+\frac{13}{2}\sin(\phi_4+\phi_5)-\frac{37}{2}\sin(\phi_5-\phi_4) \\ &+\frac{13}{2}\sin(\phi_4+\phi_6)-\frac{37}{2}\sin(\phi_6-\phi_4)\Big]+\frac{1}{5\omega+1}\Big[-3\sin(\phi_1+\phi_5)+5\sin(\phi_5-\phi_1) \\ &-3\sin(\phi_2+\phi_5)+5\sin(\phi_5-\phi_2)-\frac{3}{2}\sin(\phi_5+\phi_3)+\frac{21}{2}\sin(\phi_5-\phi_3) \\ &-\frac{3}{2}\sin(\phi_4+\phi_5)+\frac{21}{2}\sin(\phi_5-\phi_4)+\frac{1}{2}\sin(2\phi_5)+\frac{1}{2}\sin(\phi_5+\phi_6) \\ &-\frac{9}{2}\sin(\phi_6-\phi_5)\Big]+\frac{1}{5\omega-1}\Big[-3\sin(\phi_1+\phi_6)+5\sin(\phi_6-\phi_1)-3\sin(\phi_6+\phi_2) \\ &+5\sin(\phi_6-\phi_2)+\frac{21}{2}\sin(\phi_6-\phi_3)-\frac{3}{2}\sin(\phi_6+\phi_3)-\frac{3}{2}\sin(\phi_6+\phi_4) \\ &+\frac{21}{2}\sin(\phi_6-\phi_4)+\frac{1}{2}\sin(\phi_6+\phi_5)+\frac{9}{2}\sin(\phi_6-\phi_5)+\frac{1}{2}\sin(2\phi_6)\Big] \end{aligned} \tag{13}$$

$$F_{2\phi}(y,\ t)=-\frac{A^6}{(32\omega)^2}\Big\{\frac{1}{\omega+1}[80-20\cos(2\phi_1)-20\cos(\phi_1+\phi_2)$$

$$+80\cos(\phi_2-\phi_1)-60\cos(\phi_1+\phi_3)+90\cos(\phi_3-\phi_1)-60\cos(\phi_1+\phi_4)$$
$$+90\cos(\phi_4-\phi_1)-22\cos(\phi_5+\phi_1)+28\cos(\phi_5-\phi_1)-22\cos(\phi_1+\phi_6)$$
$$+28\cos(\phi_6-\phi_1)]+\frac{1}{\omega-1}[80-20\cos(\phi_1+\phi_2)+80\cos(\phi_2-\phi_1)$$
$$-20\cos(2\phi_2)-60\cos(\phi_3+\phi_2)+90\cos(\phi_3-\phi_2)-60\cos(\phi_2+\phi_4)$$
$$+90\cos(\phi_4-\phi_2)-22\cos(\phi_2+\phi_5)+28\cos(\phi_5-\phi_2)$$
$$-22\cos(\phi_2+\phi_6)+28\cos(\phi_6-\phi_2)]+\frac{1}{3\omega+1}[60+20\cos(\phi_1+\phi_3)$$
$$+70\cos(\phi_3-\phi_1)+20\cos(\phi_2+\phi_3)+70\cos(\phi_3-\phi_2)-15\cos(2\phi_3)-15\cos(\phi_3+\phi_4)$$
$$+60\cos(\phi_4-\phi_3)-8\cos(\phi_3+\phi_5)+17\cos(\phi_5-\phi_3)-8\cos(\phi_3+\phi_6)+17\cos(\phi_6-\phi_3)]$$
$$+\frac{1}{3\omega-1}[60+20\cos(\phi_4+\phi_1)+70\cos(\phi_4-\phi_1)+20\cos(\phi_2+\phi_4)+70\cos(\phi_4-\phi_2)$$
$$-15\cos(2\phi_4)-15\cos(\phi_4+\phi_3)+60\cos(\phi_4-\phi_3)-8\cos(\phi_4+\phi_5)$$
$$+17\cos(\phi_5-\phi_4)-8\cos(\phi_4+\phi_6)+17\cos(\phi_6-\phi_3)]+\frac{1}{5\omega+1}[4$$
$$+10\cos(\phi_5+\phi_1)+20(\phi_5-\phi_1)+10\cos(\phi_5+\phi_2)+20\cos(\phi_5-\phi_3)$$
$$+15\cos(\phi_5+\phi_3)+15\cos(\phi_5-\phi_4)-\cos(2\phi_5)-\cos(\phi_6+\phi_5)+4\cos(\phi_6-\phi_5)]$$
$$+\frac{1}{5\omega-1}[4+10\cos(\phi_6+\phi_1)+20\cos(\phi_6-\phi_1)+10\cos(\phi_6+\phi_2)+20\cos(\phi_6-\phi_2)$$
$$+15\cos(\phi_6-\phi_3)+15\cos(\phi_6-\phi_4)-\cos(\phi_6+\phi_5)+4\cos(\phi_6-\phi_5)$$
$$-\cos(2\phi_6)]\Big\} \tag{14}$$

4. Sub-harmonic synchronization of order 2

In this case ω is in the neighbourhood of $\frac{1}{2}$, from eq. (12) and (13) we have

$$\overline{F}_2(y,t)=\frac{A^6}{(32\omega)^2}\begin{bmatrix} A\Big[-\frac{29}{\omega+1}+\frac{5}{5\omega-1}\sin\,(\phi_6-\phi_1)+\Big(\frac{33}{\omega-1}-\frac{7}{3\omega-1}\Big)\sin(\phi_2+\phi_4)\Big] \\ -\Big[\frac{80}{\omega+1}+\frac{80}{\omega-1}+\frac{60}{3\omega+1}+\frac{60}{3\omega-1}+\frac{4}{5\omega+1}+\frac{4}{5\omega-1} \\ +\Big(\frac{28}{\omega+1}+\frac{20}{5\omega-1}\Big)\cos\,(\phi_6-\phi_1)+\Big(\frac{20}{3\omega-1}-\frac{60}{\omega-1}\Big)\cos(\phi_2+\phi_4) \end{bmatrix} \tag{15}$$

The reduced system (3) with eq. (15) takes the form:

$$\frac{d}{dt}\begin{bmatrix} A \\ \Phi \end{bmatrix} = \frac{\lambda^2 A^6}{(32\omega)^2}\begin{bmatrix} \left(-\frac{29}{\omega+1}+\frac{5}{5\omega-1}+\frac{33}{\omega-1}-\frac{7}{3\omega-1}\right)\sin(4\omega-2)t+4\phi) \\ -\left[\frac{80}{\omega+1}+\frac{80}{\omega-1}+\frac{60}{3\omega+1}+\frac{60}{3\omega-1}+\frac{4}{5\omega+1}+\frac{4}{5\omega-1}\right. \\ \left.+\left(\frac{28}{\omega+1}+\frac{20}{5\omega-1}+\frac{20}{3\omega-1}-\frac{60}{\omega-1}\right)\cos((4\omega-2)t+4\Phi)\right] \end{bmatrix} \tag{16}$$

For obtaining sub-harmonic synchronization of order 2, we must have,

$$\frac{dA}{dt}=0 \tag{17}$$

To satisfy this condition, we must take

$$(4\omega-2)t+4\Phi=k\pi \tag{18}$$

Where $k=0, 1, 2, 3, 4, 5, 6, 7.$ with the aid eq. (18) and the second equation of (16), we obtain

$$\omega^2=\frac{1}{4}+0.91\ \lambda^2A^6 \tag{19}$$

for $k=0, 2, 4, 6$, and

$$\omega^2=\frac{1}{4}-0.58\ \lambda^2A^6 \tag{20}$$

for $k=1, 3, 5, 7$

Then we obtain 8 sub-harmonic synchronization of order 2, 4 for $\omega^2<\frac{1}{4}$ and 4 for $\omega^2>\frac{1}{4}$ from the first function of eq. (7) and (11), we have

$$\tilde{G}_1=(y,\ t)=\begin{bmatrix} \tilde{G}_{1a}(y,\ t) \\ \tilde{G}_{1\varphi}(y,\ t) \end{bmatrix} \tag{21}$$

Also, from the second function of eq. (7) and (11), we have

$$\tilde{G}_2(y,\ t)=\begin{bmatrix} \tilde{G}_{2a}(y,\ t) \\ \tilde{G}_{2\phi}(y,\ t) \end{bmatrix}$$

The amplitude and the total phase for the second approximation are defined by:

$$a(t)=A+\lambda\tilde{G}_{1a}(y,\ t)+\lambda^2\tilde{G}_{2a}(y,\ t),\quad \phi=\omega t+\phi+\lambda\tilde{G}_{1\phi}(y,\ t)+\lambda^2\tilde{G}_{2\phi}(y,\ t) \tag{23}$$

By substitution from eq. (21) and (22) into (23), one obtains

$$a(t)=A-\frac{\lambda A^4}{16}\left[\frac{2}{7}\cos\left(\frac{7}{2}t+\frac{5k\pi}{4}\right)+\frac{6}{5}\cos\left(\frac{5}{2}t+\frac{3k\pi}{4}\right)+\frac{2}{3}\cos\left(\frac{3}{2}t+\frac{5k\pi}{4}\right)\right.$$

$$+\frac{4}{3}\cos\left(\frac{3}{2}t+\frac{k\pi}{4}\right)+6\cos\left(\frac{t}{2}+\frac{3k\pi}{4}\right)-4\cos\left(\frac{t}{2}-\frac{k\pi}{4}\right)\Big]$$

$$-\frac{\lambda^2A^7}{256}\Big[\frac{1}{49}\cos\left(7t+\frac{5}{2}k\pi\right)+\frac{38}{105}\cos(6t+2k\pi)+\frac{2}{21}\cos\left(5t+\frac{5}{2}k\pi\right)$$

$$+\frac{523}{175}\cos\left(5t+\frac{3}{2}k\pi\right)+\frac{124}{35}\cos(4t+2k\pi)+\frac{688}{35}\cos(4t+k\pi)$$

$$+\frac{1}{9}\cos\left(3t+\frac{5}{2}k\pi\right)-\frac{702}{105}\cos\left(3t+\frac{3}{2}k\pi\right)+\frac{1127}{45}\cos\left(3t+\frac{k\pi}{2}\right)$$

$$+6\cos(2t+2k\pi)-\frac{304}{35}\cos(2t+k\pi)-\frac{1593}{70}\cos(2t)-35\cos\left(t+\frac{3}{2}k\pi\right)$$

$$+\frac{156}{15}\cos\left(t+\frac{k\pi}{2}\right)-\frac{338}{5}\cos\left(t-\frac{k\pi}{2}\right)\Big]. \tag{24}$$

$$\Psi=\omega t+\Phi+\frac{\lambda A^3}{16}\Big[\frac{2}{7}\sin\left(\frac{7}{2}t+\frac{5k\pi}{4}\right)+2\sin\left(\frac{5}{2}t+\frac{3k\pi}{4}\right)+\frac{2}{3}\sin\left(\frac{3}{2}t+\frac{5k\pi}{4}\right)$$

$$+\frac{20}{3}\sin\left(\frac{3}{2}t+\frac{k\pi}{4}\right)+10\sin\left(\frac{t}{2}+\frac{3k\pi}{4}\right)+20\sin\left(\frac{t}{2}-\frac{k\pi}{4}\right)\Big]$$

$$+\frac{\lambda^2A^6}{256}\Big[\frac{2}{49}\sin\left(7t+\frac{5}{2}k\pi\right)-\frac{19}{105}\sin(6t+2k\pi)+\frac{4}{21}\sin\left(5t+\frac{5}{2}k\pi\right)$$

$$+\frac{374}{105}\sin\left(5t+\frac{3}{2}k\pi\right)+\frac{24}{5}\sin(4t+2k\pi)+22\sin(4t+2k\pi)$$

$$+\frac{2}{9}\sin\left(3t+\frac{5}{2}k\pi\right)+\frac{20}{21}\sin\left(3t+\frac{3}{2}k\pi\right)+\frac{2311}{63}\sin\left(3t+\frac{k\pi}{2}\right)$$

$$+8\sin(2t+2k\pi)-\frac{384}{7}\sin(2t+k\pi)-\frac{532}{21}\sin(2t)-\frac{62}{3}\sin\left(t+\frac{3}{2}k\pi\right)$$

$$-\frac{13774}{105}\sin\left(t+\frac{k\pi}{2}\right)-\frac{1284}{5}\sin\left(t-\frac{k\pi}{2}\right)\Big]. \tag{25}$$

Then the approach analytical expression is:

$$x=a(t)\cos(\omega t+\phi(t))=[A+\lambda\tilde{G}_{1a}(y,t)+\lambda^2\tilde{G}_{2a}(y,t)]\cos(\omega t+\Phi+\lambda\tilde{G}_1(y,t)+\lambda^2\tilde{G}_2(y,t))$$

By using Talyor expansion for the second approximation of λ, we have

$$x=A\cos\left(\frac{t}{2}+\frac{k\pi}{4}\right)-\frac{\lambda A^4}{32}\Big[\frac{1}{7}\cos\left(4t+\frac{3}{2}k\pi\right)+\frac{22}{35}\cos(3t+k\pi)$$

$$+\frac{1}{3}\cos\left(2t+\frac{3}{2}k\pi\right)+\frac{1}{5}\cos\left(2t+\frac{k\pi}{2}\right)+2\cos(t+k\pi)-\frac{28}{3}\cos(t)$$

$$+17\cos\left(\frac{k\pi}{2}\right)\Big]-\frac{\lambda^2A^7}{(32)^2}\Big[\frac{2}{49}\cos\left(\frac{15}{2}t+\frac{11}{4}k\pi\right)+\frac{281}{315}\cos\left(\frac{13}{2}t+\frac{9}{4}k\pi\right)$$

$$+\frac{4}{21}\cos\left(\frac{11}{2}t+\frac{11}{4}k\pi\right)+\frac{209}{75}\cos\left(\frac{11}{2}t+\frac{7}{4}k\pi\right)+\frac{176}{35}\cos\left(\frac{9}{2}t+\frac{9}{4}k\pi\right)$$

$$+\frac{13928}{595}\cos\left(\frac{9}{2}t+\frac{5}{4}k\pi\right)+\frac{1}{3}\cos\left(\frac{7}{2}t+\frac{11}{4}k\pi\right)+\frac{266}{105}\cos\left(\frac{7}{2}t+\frac{7}{4}k\pi\right)$$

$$+\frac{19097}{315}\cos\left(\frac{7}{2}t+\frac{3}{4}k\pi\right)+\frac{86}{9}\cos\left(\frac{5}{2}t+\frac{9}{4}k\pi\right)+\frac{628}{5}\cos\left(\frac{5}{2}t+\frac{5}{4}k\pi\right)$$
$$+\frac{28576}{315}\cos\left(\frac{5}{2}t+\frac{k\pi}{4}\right)-\frac{134}{3}\cos\left(\frac{3}{2}t+\frac{7}{4}k\pi\right)+\frac{2127}{35}\cos\left(\frac{3}{2}t+\frac{3}{4}k\pi\right)$$
$$+\frac{277666}{75}\cos\left(\frac{3}{2}t-\frac{k\pi}{4}\right)+\frac{1750}{9}\cos\left(\frac{t}{2}+\frac{5}{4}k\pi\right)-\frac{4811}{35}\cos\left(\frac{t}{2}+\frac{k\pi}{4}\right)$$
$$-\frac{13048}{45}\cos\left(\frac{t}{2}-\frac{3}{4}k\pi\right)\Big]$$

when $\lambda\to O$, $a(t)\to A$, $\frac{d\Psi}{dt}\to 1/2$, and $x\to A\cos\left(\frac{t}{2}+\frac{k\pi}{4}\right)$.

5. Sub-harmonic synchroization of order 4

When ω is in the neighbourhood of $\frac{1}{4}$, we get what is called sub-harmonic synchronization of order 4, from eq. (13) and (4), we get the function $\overline{F}_2(y,t)$, then the reduced system (3) takes the form:

$$\frac{d}{dt}\begin{bmatrix}A\\ \Phi\end{bmatrix}=\frac{\lambda^2A^6}{(32\omega)^2}\begin{bmatrix}A\left[1/2\left(\frac{13}{3\omega-1}-\frac{3}{5\omega-1}\right)\sin((8\omega-2)t+8\Phi)\right.\\ -\left[\frac{80}{\omega+1}+\frac{80}{\omega-1}+\frac{60}{3\omega+1}+\frac{60}{3\omega-1}+\frac{4}{5\omega+1}+\frac{4}{5\omega-1}\right.\\ \left.-\frac{8}{3\omega-1}\cos((8\omega-2)t+8\Phi)\right]\end{bmatrix} \tag{26}$$

To satisfy (17), we must take

$$(8\omega-2)t+8\Phi=k\pi \tag{27}$$

where $k=0, 1, 2, 3, \ldots, 15$.

With the aid of eq. (27) and the second function of eq. (26), we obtain

$$\omega^2=\frac{1}{16}-1.55\lambda^2A^6,\ \text{for } k=0, 2, 4, \ldots, 14 \tag{28}$$

and

$$\omega^2=\frac{1}{16}-2.05\lambda^2A^6,\ \text{for } k=1, 3, \ldots, 15 \tag{29}$$

Then we obtain 16 sub-harmonic synchronization of order 4, for $\omega^2<\frac{1}{16}$.

From the eq. (7) and (11), we have

$$\tilde{G}_1(y,t)=\begin{bmatrix}\tilde{G}_{1a}(y,t)\\ \tilde{G}_{1\phi}(y,t)\end{bmatrix},\quad \tilde{G}_2(y,t)=\begin{bmatrix}\tilde{G}_{2a}(y,t)\\ \tilde{G}_{2\phi}(y,t)\end{bmatrix}. \tag{30}$$

By substitution from (30) into (23), we obtain

$$a(t)=A-\frac{\lambda A^4}{8}\Big[\frac{4}{9}\cos\left(\frac{9}{4}t+\frac{5}{8}k\pi\right)+\frac{12}{7}\cos\left(\frac{7}{4}t+\frac{3k\pi}{8}\right)$$

$$+\frac{8}{5}\cos\left(\frac{5}{4}t+\frac{k\pi}{8}\right)-\frac{8}{3}\cos\left(\frac{3}{4}t-\frac{k\pi}{8}\right)+4\cos\left(\frac{t}{4}t+\frac{5}{8}k\pi\right)$$

$$-12\cos\left(\frac{t}{4}+\frac{3}{8}k\pi\right)\Bigg]-\frac{\lambda^2A^7}{64}\Bigg[\frac{4}{81}\cos\left(\frac{9}{2}t+\frac{5}{4}k\pi\right)+\frac{16}{21}\cos(4t+k\pi)$$

$$+\frac{3578}{735}\cos\left(\frac{7}{2}t+\frac{3k\pi}{4}\right)+\frac{2848}{135}\cos\left(3t+\frac{k\pi}{2}\right)+\frac{8}{9}\cos\left(\frac{5}{2}t+\frac{5}{4}k\pi\right)$$

$$+\frac{34856}{525}\cos\left(\frac{5}{2}t+\frac{k\pi}{4}\right)+\frac{1123}{21}\cos\left(3t+\frac{3737}{35}\right)+\frac{491}{18}\cos\left(\frac{3}{2}t+\frac{3}{4}k\pi\right)$$

$$+\frac{5998}{105}\cos\left(\frac{3}{2}t-\frac{k\pi}{4}\right)+\frac{1984}{45}\cos\left(t+\frac{k\pi}{2}\right)+\frac{96}{5}\cos\left(t-\frac{k\pi}{2}\right)+4\cos\left(\frac{t}{2}+\frac{5}{4}k\pi\right)$$

$$+\frac{1136}{7}\cos\left(\frac{t}{2}+\frac{k\pi}{4}\right)+116\cos\left(\frac{t}{2}-\frac{3k\pi}{4}\right)\Bigg] \qquad (31)$$

$$\Psi=t+\Phi+\frac{\lambda A^3}{8}\Bigg[\frac{4}{9}\sin\left(\frac{9}{4}t+\frac{5}{8}k\pi\right)+\frac{20}{7}\sin\left(\frac{7}{4}t+\frac{3}{8}k\pi\right)$$

$$+8\sin\left(\frac{5}{4}t+\frac{k\pi}{8}\right)-\frac{40}{3}\sin\left(\frac{3}{4}t-\frac{k\pi}{8}\right)+4\sin\left(\frac{t}{4}+\frac{5k\pi}{8}\right)-20\sin\left(\frac{t}{4}-\frac{3}{8}k\pi\right)\Bigg]$$

$$-\frac{\lambda^2A^6}{64}\Bigg[\frac{-8}{81}\sin\left(\frac{9}{2}t+\frac{5}{4}k\pi\right)+\frac{11}{21}\sin(4t+k\pi)+\frac{2568}{245}\sin\left(\frac{7}{2}t+\frac{3}{4}k\pi\right)$$

$$-\frac{176}{9}\sin\left(3t+\frac{k\pi}{2}\right)-\frac{16}{9}\sin\left(\frac{5}{2}t+\frac{5}{4}k\pi\right)-\frac{4.4}{15}\sin\left(\frac{5}{2}t+\frac{k\pi}{4}\right)$$

$$+\frac{96}{7}\sin(2t+k\pi)-\frac{7264}{63}\sin(2t)+\frac{1856}{35}\sin\left(\frac{3}{2}t+\frac{3}{4}k\pi\right)$$

$$-\frac{4688}{63}\sin\left(\frac{3}{2}t-\frac{k\pi}{4}\right)-\frac{11776}{315}\sin\left(t+\frac{k\pi}{2}\right)+\frac{512}{5}\sin\left(t-\frac{k\pi}{2}\right)$$

$$-8\sin\left(\frac{t}{2}+\frac{5}{4}k\pi\right)-\frac{25042}{63}\sin\left(\frac{t}{2}+\frac{k\pi}{4}\right)-\frac{416}{3}\sin\left(\frac{t}{2}-\frac{3}{4}k\pi\right)\Bigg] \qquad (32)$$

Then the approach analytical expression for subharmonic synchronization or order 4 is:

$$x=A\cos\left(\frac{t}{4}+\frac{k\pi}{8}\right)+\frac{\lambda A^4}{8}\Bigg[\frac{16}{63}\cos\left(2t+\frac{k\pi}{2}\right)+\frac{64}{35}\cos\left(\frac{3}{2}t+\frac{k\pi}{4}\right)$$

$$-\frac{304}{15}\cos(t)+8\cos\left(\frac{t}{2}-\frac{k\pi}{4}\right)+24\cos\left(\frac{k\pi}{2}\right)\Bigg]-\frac{\lambda^2A^7}{256}\Bigg[\frac{11}{81}\cos\left(\frac{19}{4}t+\frac{11}{8}k\pi\right)$$

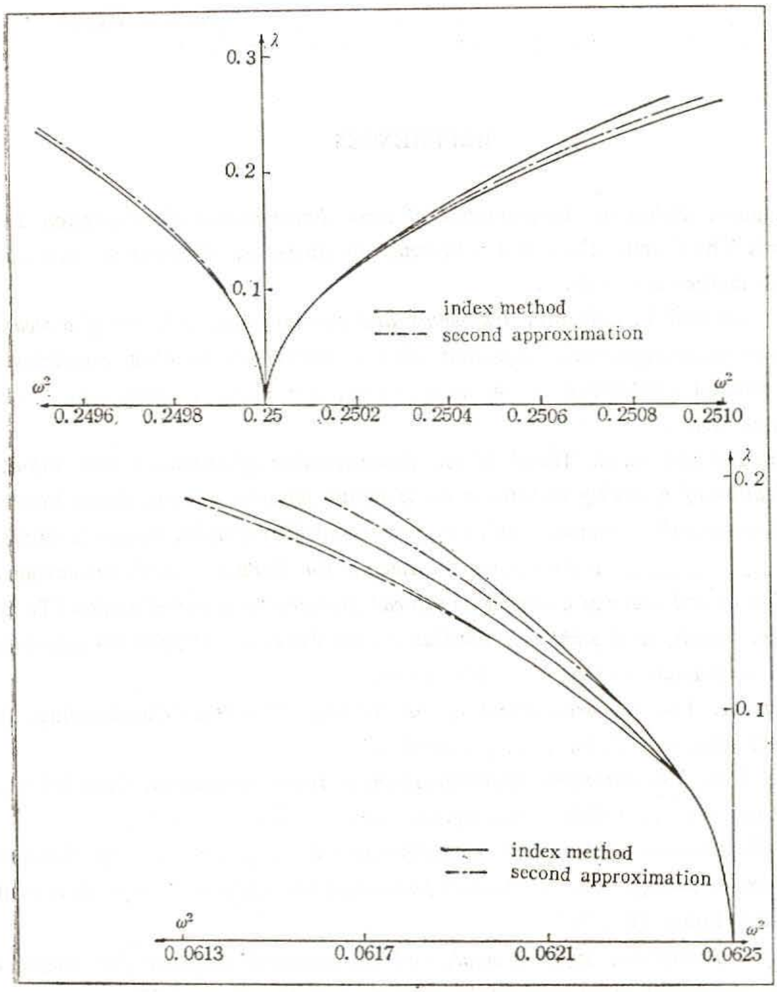
$$+\frac{9793}{4536}\cos\left(\frac{17}{4}\mathrm{t}+\frac{9}{8}k\pi\right)+\frac{38481}{17640}\cos\left(\frac{15}{2}t+\frac{7}{8}k\pi\right)+\frac{739047}{6615}\cos\left(\frac{13}{4}t+\frac{5}{8}k\pi\right)$$

$$+\frac{43}{18}\cos\left(\frac{11}{4}t+\frac{11}{8}k\pi\right)+\frac{17549}{11025}\cos\left(\frac{11}{4}t+\frac{3}{8}k\pi\right)-\frac{6166}{63}\cos\left(\frac{9}{4}t+\frac{9}{8}k\pi\right)$$

$$+\frac{119053}{70}\cos\left(\frac{9}{4}t+\frac{k\pi}{8}\right)+\frac{7997}{3780}\cos\left(\frac{7}{4}t+\frac{7}{8}k\pi\right)+\frac{11806}{35}\cos\left(\frac{7}{4}t-\frac{k\pi}{8}\right)$$

$$-\frac{712}{63}\cos\left(\frac{5}{4}t+\frac{5}{8}k\pi\right)+\frac{81412}{315}\cos\left(\frac{5}{4}t-\frac{3}{8}k\pi\right)+11\cos\left(\frac{3}{4}t+\frac{11}{8}k\pi\right)$$

$$+\frac{112103}{315}\cos\left(\frac{3}{4}t+\frac{3k\pi}{8}\right)-\frac{4664}{15}\cos\left(\frac{3}{4}t-\frac{5k\pi}{8}\right)+13\cos\left(\frac{t}{4}+\frac{9}{8}k\pi\right)$$
$$+\frac{2583901}{6615}\cos\left(\frac{t}{4}+\frac{k\pi}{8}\right)+\frac{920}{3}\cos\left(\frac{t}{4}-\frac{7}{8}k\pi\right)\Big]$$



(Fig. d) Results obtained by two Methods

When $\lambda \to O$, $a(t) \to A$, $\frac{d\Psi}{dt} \to \frac{1}{4}$ and $x \to A\cos\left(\frac{t}{4}+\frac{k\pi}{8}\right)$.

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