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RINGS OF BOUNDED REAL-VALUED FUNCTIONS AND THE FINITE SUBCOVERING PROPERTY

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In what follows all the functions are real-valued and 0, 1 stand for the zero and the unit real numbers respectively. Also, all the ring-theoretical statements which are made in connection with a set of functions (with a common domain) refer to the pointwise addition and multiplication of the elements of that set. Moreover, if f is a function then we let $|f|$, as usual, stand for the function whose values are the absolute values of f.

Let c be a function whose domain is X. Then, as expected, c is called a *constant* function on X if and only if $c(x)=r$ for every $x\in X$. In particular, if $r=1$ then c is called the *unit function* on X.

Let E' be a set of functions with a common domain X' and let u' be the unit function of X'. We say that u' is covered by the elements of E' if and only if for every $z\in X'$ we have:

(1) $$f'(z)\neq 0 \text{ for some } f'\in E'$$

LEMMA. *Let F' be a finite subset of a ring C' of functions with a common domain X' and let $u'\in C'$ where u' is the unit function on X'. If u' is not covered by the elements of F' then the ideal J' of C' generated by F' is proper.*

PROOF. Since u' is not covered by the elements of F' from (1) it follows that for some $z\in X'$ it is the case that $f'(z)=0$ for every $f'\in F'$. But then in $g'\in J'$ we see that $g'(z)=0$ and therefore $g'\neq u'$. Hence $u'\notin J'$ and J' is a proper ideal.

From the Lemma we have immediately:

COROLLARY. *Let C' be a ring as mentioned in the Lemma and E' be a subset of C'. If u' is covered by no finite number of elements of E' then the ideal I' of C' generated by E' is proper.*

Based on the above, we have:

THEOREM. *Let $C=\{u, \dots, f, g, \dots\}$ be a ring of bounded functions with a common domain X such that every constant function on X is an element of C and if $f\in C$ then $|f|\in C$ and where u is the unit function on X. Then, X can be* *extended to a set X' and every element f of C can be extended to a function f' with X' as its domain such that:*

(i) *the resulting set $C' = \{u', \dots, f', g', \dots\}$ of extended functions is a ring.*

(ii) *the correspondence $f \longrightarrow f'$ is a ring isomorphism from C onto C'.*

(iii) *if u' is covered by the elements of a subset E' of C' then u' is already covered by finitely many elements of E'.*

PROOF. Since C contains all the constant functions on X, the set M_x given by:

(2) $$M_x = \{f \mid (f \in C) \text{ and } f(x) = 0\} \text{ for every } x \in X$$

is a real ideal of C (i.e., C/M_x is isomorphic to the reals).

Let

(3) $$\{M_y \mid y \in Y\}$$

be the set of all the real ideals M_y of C such that $M_y \notin \{M_x \mid x \in X\}$ where M_x is given by (2).

Let us consider the set X' given by:

(4) $$X' = X \cup Y$$

where Y is as in (3).

Clearly, X' is an extension of X. Moreover, form (2), (3), (4) it follows that:

(5) $$\{M_z \mid z \in X'\}$$

is the set of all the real ideals of C.

To every $f \in C$ let us correspond a function f' on X' defined as:

(6) $$f' = f \text{ on } X \text{ and } f'(y) = r = f \text{ Mod } M_y \text{ for every } y \in Y.$$

Obviously, f' is an extension of f.

From (2), (3), (6) we have:

$$M_y = \{f \mid (f \in C) \text{ and } f'(y) = 0\} \text{ for every } y \in Y$$

which by (2), (4) implies that

(7) $$M_z = \{f \mid (f \in C) \text{ and } f'(z) = 0\} \text{ for every } z \in X'.$$

Again, from (6) it readily follows that:

(8) $$f' + g' = (f+g)' \text{ and } f'g' = (fg)'$$

and since C is a ring, we see that the set C' given by

(9) $$C' = \{f' \mid f \in C\} = \{u', \dots, f', g', \dots\}$$

is also ring. Hence (i) is established.

Clearly, by (5) if $f' = g'$ then $f = g$ which by (8) implies that the correspondence

$f \longrightarrow f'$ is a ring isomorphism from C onto C'. Hence (ii) is also established.

Next, let the unit (on X') function u' be covered by the elements of a subset E' of C', i.e., as in (1), for every $z \in X'$ we have:

(10) $$f'(z) \neq 0 \text{ for some } f' \in E'.$$

To prove (iii) we must show that u' is already covered by some finitely many elements of E'. Let us assume to the contrary that u' is covered by no finite number of elements of E'. Thus, by the Corollary, the ideal I' of C' generated by E' is proper. Obviously,

(11) $$E' \subseteq I'$$

Since u' is the extension of u and $f \longrightarrow f'$ is a ring isomorphism, we see that the subset I of C defined by:

(12) $$I = \{f \mid f' \in I'\}$$

is a proper ideal of C. Therefore I is contained in a maximal ideal M of C. Let us observe that C is a lattice where $f \vee g$ and $f \wedge g$ are equal respectively to $\frac{1}{2}(f+g+|f-g|)$ and $\frac{1}{2}(f+g-|f-g|)$. Thus, by [1, p. 66] the maximal ideal M is an absolutely convex ideal of C and therefore C/M is a totally ordered field. Also, since every $f \in C$ is bounded, C/M is Archimedean. Moreover, since every constant function is an element of C we see that C/M has a subfield isomorphic to the reals. But then, as such, C/M itself is isomorphic to the reals. Consequently, M is a real ideal of C and, in view of (5), we have:

(13) $$I \subseteq M = M_z \text{ for some } z \in X'$$

Now, if $f' \in E'$ then from (11), (12), (13), (7) it follows that

$$f'(z) = 0 \text{ for every } f' \in E'$$

which contradicts (10). Hence our assumption is false and (iii) is established.

REMARK. We observe that X (as well as X') can be topologized with subbasic open sets of the form $\{x \mid f(x) \neq 0\}$ for some $f \in C$ (as well as for some $f' \in C'$) which, in fact form a base. Moreover, we can verify that C' includes all the constant functions and if $f' \in C'$ then $|f'| \in C'$, as in the case of C. From this it follows that all the elements of C (as well as of C') are continuous functions (with the reals topologized as usual). Obviously, if the points of X are separated by the elements of C then X becomes completely regular Hausdorff. Clearly, if C is the ring of all bounded continuous functions then X is C^*-embedded in X' and hence X' is the Stone-Čech compactification of X.

The above Remark shows that the topological structure involved in the

Stone-Čech compactification can be fully recovered from the underlying algebraic structure.

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