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A REMARK ON THE SPECIAL CLASSES OF ANALYTIC FUNCTIONS II

By Shigeyoshi Owa

Introduction

There are many classes of analytic functions in the unit disk U. In this place, we consider about the special classes $C(k)$, $P^*(\alpha, \beta)$, $D_0(k)$, $G(k)$, $P(k)$, and $R(k)$ of analytic functions in the unit disk U. And it is the purpose of this paper to give a relation among these classes.

DEFINITION 1. Let $D(k)$ denote the class of functions

$$f(z)=z+\sum_{n=2}^{\infty} a_n z^n$$

analytic in the unit disk U and satisfying

$$\left|\frac{f'(z)-1}{f'(z)+1}\right|<k \ (z\in U)$$

for some $k(0<k\leqq 1)$. And let $D_0(k)$ denote the class of analytic and univalent functions $f(z)$ in the class $D(k)$.

For this class $D(k)$, K.S. Padmanabhan showed the following result in [4].

LEMMA 1. *Let the function*

$$f(z)=z+\sum_{n=2}^{\infty} a_n z^n$$

belong to the class $D(k)$. *Then, we have*

$$|f'(z)|\leqq\frac{1+k|z|}{1-k|z|}$$

for $z\in U$.

Moreover, S. Owa showed some results for the fractional calculus of functions $f(z)$ in this class $D(k)$ in [2].

DEFINITION 2. Let $R(k)$ denote the class of functions

$$f(z)=z+\sum_{n=2}^{\infty} a_n z^n$$

analytic in the unit disk U and satisfying

$$\mathrm{Re}\{f'(z)\}>k$$

for some $k(0\leqq k<1)$.

For this class $R(k)$, D.B. Shaffer gave some results in [5].

DEFINITION 3. Let $P(k)$ denote the class of functions

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n$$

analytic in the unit disk U and satisfying

$$\left| f'(z)-\frac{1}{2k} \right| \leqq \frac{1}{2k}$$

for some $k(0\leqq k<1)$.

For this class $P(k)$, D.B. Shaffer showed the following lemma in [6].

LEMMA 2. *Let the function*

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n$$

be in the class $P(k)$. *Then, for* $z\in U$,

$$\frac{1-|z|}{1+(1-2k)|z|}\leqq\mathrm{Re}\{f'(z)\}\leqq\frac{1+|z|}{1-(1-2k)|z|}.$$

DEFINITION 4. Let $G(k)$ denote the class of functions

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n$$

analytic and univalent in the unit disk U and satisfying

$$|f'(z)-1|<k$$

for some $k(0\leqq k\leqq 1)$.

For this class $G(k)$, V. Singh gave some results in [8].

DEFINITION 5. Let $C(k)$ denote the class of functions

$$f(z)=z-\sum_{n=2}^{\infty}a_n z^n \quad (a_n\geqq 0)$$

that are convex of order $k(0\leqq k<1)$ in the unit disk U.

For this class $C(k)$, H. Silverman gave some results in [7].

DEFINITION 6. Let $P^*(\alpha, \beta)$ denote the class of functions

$$f(z)=z-\sum_{n=2}^{\infty}a_n z^n \quad (a_n\geqq 0)$$

analytic and univalent in the unit disk U for which

$$\left| \frac{f'(z)-1}{f'(z)+(1-2\alpha)} \right| < \beta,$$

where $0 \leqq \alpha < 1$ and $0 < \beta \leqq 1$.

For this class $P^*(\alpha, \beta)$, V.P. Gupta and P.K. Jain showed some results in [1].

Recently, S. Owa show the following lemma in [3].

LEMMA 3. *Let* $0 < k \leqq 1$. *Then, we have*

$$C(1-k) \subset P^*(0, k) \subset D(k) \subset R\left(\frac{1-k}{1+k}\right).$$

In particular,

$$C(2-\sqrt{2}) \subset P^*(0, \sqrt{2}-1) \subset D(\sqrt{2}-1) \subset R(\sqrt{2}-1)$$

and

$$C(0) \subset P^*(0, 1) \subset D(1) \subset R(0).$$

THEOREM. *Let* $0 < k \leqq 1/3$. *Then, we have*

$$C(1-k) \subset P^*(0, k) \subset D_0(k) \subset G\left(\frac{2k}{1-k}\right) \subset P\left(\frac{1}{2}\right) \subset R(0).$$

PROOF. In the first place, let a function

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

be in the class $D_0(k)$. Then, we have

$$|f'(z)| \leqq \frac{1+k}{1-k}$$

with the aid of Lemma 1. Moreover, from Definition 1,

$$|f'(z)-1| < k(|f'(z)|+1).$$

Hence, we have

$$|f'(z)-1| < k\left(\frac{1+k}{1-k}+1\right) = \frac{2k}{1-k}.$$

Consequently, we have

$$D_0(k) \subset G\left(\frac{2k}{1-k}\right)$$

for $0 < k \leqq 1/3$.

In the second place, we have

$$G\left(\frac{2k}{1-k}\right) \subset P\left(\frac{1}{2}\right)$$

for $0<k\leqq 1/3$ from Definition 3 and Definition 4. And if the function $f(z)$ belongs to the class $P(1/2)$, by using Lemma 2, we have briefly

$$\mathrm{Re}\{f'(z)\}>0,$$

that is,

$$P\left(\frac{1}{2}\right)\subset R(0).$$

Accordingly, we have the relation

$$C(1-k)\subset P^*(0,\ k)\subset D_0(k)\subset G\left(\frac{2k}{1-k}\right)\subset P\left(\frac{1}{2}\right)\subset R(0)$$

for $0<k\leqq 1/3$ with the aid of Lemma 3.

COROLLARY. *We have the relation*

$$C\left(\frac{2}{3}\right)\subset P^*\left(0,\ \frac{1}{3}\right)\subset D_0\left(\frac{1}{3}\right)\subset G(1)\subset P\left(\frac{1}{2}\right)\subset R(0).$$

Kinki Univ.
Osaka, Japan

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