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ON SPACES NOWHERE LOCALLY COMPACT

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1. Introduction

DEFINITION 1.1. A space $(X, \mathscr{T})$ will be termed *nowhere locally compact* (written henceforth as *nlc*) iff $Int K=\phi$ for all compact sets K, Int denoting the interior operator.

We begin with a few examples of *nlc* spaces.

EXAMPLE 1.2. (H, d) where H consists of all infinite sequences of reals $(x_1, x_2, \dots)$ for which $\Sigma x_i^2<\infty$ and $d(x, y)=(\Sigma|x_i-y_i|^2)^{1/2}$.

EXAMPLE 1.3. $(X, \mathscr{T})$ where X is uncountable and $\mathscr{T}$ is the cocountable topology. (The only compact sets in X are finite sets.)

EXAMPLE 1.4. $(X, \mathscr{T})$ where X is the set of rational numbers and $\mathscr{T}$ is the usual topology.

EXAMPLE 1.5. $(X, \mathscr{T})$ where X is the set of reals and $\mathscr{T}$ is the half open interval topology.

EXAMPLE 1.6. $X\{(X_\alpha, \mathscr{T}_\alpha) : \alpha\in\Delta\}$ where $(X_\alpha, \mathscr{T}_\alpha)$ is not compact for an infinite number of α. (See Theorem 16, page 145 in [1].)

$\mathscr{C}$ will denote the complement operator and c will denote the closure operator.

If $(X, \mathscr{T})$ is a *nlc* space, we will say that $\mathscr{T}$ is a *nlc* topology or simply that $\mathscr{T}$ is *nlc*.

2. Subspaces of *nlc* spaces

THEOREM 2.1. *Let $(X, \mathscr{T})$ be a nlc space and O an nonempty open subset of X. Then $(O, O\cap\mathscr{T})$ is nlc.*

THEOREM 2.2. *Let $(Y, \mathscr{U})$ be a dense subspace of a Hausdorff nlc space $(X, \mathscr{T})$. Then $(Y, \mathscr{U})$ is nlc.*

PROOF. Suppose $\phi\neq O\cap Y\subseteq K\subseteq Y$ where $O\in\mathscr{T}$ and K is compact. Then $\phi\neq O\subseteq c_X(O)=c_X(O\cap Y)=c_X(K)=K$. Thus $\phi\neq O\subseteq K\subseteq X$, and X is not *nlc*, a contradiction.

THEOREM 2.3. *Suppose* $Y \subseteq X$ *and* $(X, \mathscr{T})$ *is a nlc space. If* $\complement Y$ *is compact, then* $(Y, Y \cap \mathscr{T})$ *in nlc.*

PROOF. Suppose $\phi \neq O \cap Y \subseteq K \subseteq Y$ where $O \in \mathscr{T}$ and K is compact. Then $\phi \neq O \subseteq K \cup \complement Y \subseteq X$. Then X is not *nlc*, a contradiction.

REMARK 2.4. A closed subset of a *nlc* space need not be *nlc*, e.g., a singleton set in the space of rationals is not *nlc*.

3. Product spaces

THEOREM 3.1. *Let* $(X, \mathscr{T}) = X\{(X_\alpha, \mathscr{T}_\alpha) : \alpha \in \Delta\}$ *where* $\mathscr{T}$ *is the product topology or the box topology. Suppose* $(X_\beta, \mathscr{T}_\beta)$ *is nlc for some* $\beta \in \Delta$. *Then* $(X, \mathscr{T})$ *is nlc.*

PROOF. Suppose on the contrary that $\phi \neq O \subseteq K \subseteq X$ where $O \in \mathscr{T}$ and K is compact. Then $\phi \neq P_\beta[O] \subseteq P_\beta[K] \subseteq X_\beta$. Then $P_\beta[O]$ is open and $P_\beta[K]$ is compact, P_β denoting the β-projection. Thus X_β is not *nlc*, a contradiction.

The converse of Theorem 3.1 is false as seen in

EXAMPLE 3.2. Let $(X_n, \mathscr{T}_n) = (R, \mathscr{U})$ for $n \geq 1$ where $(R, \mathscr{U})$ is the space of reals with the usual topology. If $(X, \mathscr{T}) = X\{(X_n, \mathscr{T}_n) : n \geq 1\}$, then $(X, \mathscr{T})$ is *nlc* by Example 1.5. However $(X_n, \mathscr{T}_n)$ is *nlc* for no integer n.

4. Enlarging *nlc* topologies

THEOREM 4.1. *Let* $(X, \mathscr{T})$ *be nlc and let* $\mathscr{U}$ *be the cofinite topology on* X. *Then* $sup\{\mathscr{T}, \mathscr{U}\}$ *is a nlc topology for* X.

PROOF. Suppose $\phi \neq A \subseteq K \subseteq X$ where A is in $sup\{\mathscr{T}, \mathscr{U}\}$ and K is compact relative to $sup\{\mathscr{T}, \mathscr{U}\}$. Then there exists a $U \in \mathscr{U}$ and an $O \in \mathscr{T}$ such that $\phi \neq O \cap U \subseteq K \subseteq X$. It follows that $\phi \neq O \subseteq K \cup \complement U \subseteq X$. But $K \cup \complement U$ is $\mathscr{T}$-compact and thus $(X, \mathscr{T})$ is not *nlc*, a contradiction.

THEOREM 4.2. *Let* $(X, \mathscr{T})$ *be nlc and* $K \subseteq X$, K *compact. If* $\mathscr{U} = \{\phi, \complement K, X\}$, *then* $sup\{\mathscr{T}, \mathscr{U}\}$ *is a nlc topology for* X.

PROOF. Modify the proof of Theorem 4.1.

THEOREM 4.3. *Let* $(X, \mathscr{T})$ *be nlc and* $c(A) = X$. *If* $\mathscr{U} = \{\phi, A, X\}$ *and every compact set in* $(X, \mathscr{T})$ *is closed, then* $sup\{\mathscr{T}, \mathscr{U}\}$ *is a nlc topology for* X.

PROOF. Let $\phi \neq D \subseteq K \subseteq X$ where D is in $sup\{\mathscr{T}, \mathscr{U}\}$ and K is compact relative to $sup\{\mathcal{T}, \mathcal{U}\}$. Then there exist an $O\in\mathcal{T}$ such that $\phi\neq O\cap A\subseteq D\subseteq K\subseteq X$. Then $\phi\neq O\subseteq c(O)=c(O\cap A)\subseteq c(K)=K$. Thus $\phi\neq O\subseteq K\subseteq X$ and $(X, \mathcal{T})$ is not *nlc*, a contradiction.

THEOREM 4.4. *Let $(X, \mathcal{T})$ be nlc and $O^*\subseteq A\subseteq c(O^*)$ for some $O^*\in\mathcal{T}$. Then $sup\{\mathcal{T}, \{\phi, A, X\}\}$ is nlc.*

PROOF. Let $\phi\neq U\subseteq K\subseteq X$ where $U\in sup\{\mathcal{T}, \{\phi, A, X\}\}$ and K is compact relative to $sup\{\mathcal{T}, \{\phi, A, X\}\}$. We may assume that $\phi\neq O\cap A\subseteq U$ for some $O\in\mathcal{T}$. It follows that $\phi\neq O\cap O^*\subseteq K\subseteq X$ and thus $(X, \mathcal{T})$ is not *nlc*, a contradiction.

5. When is a topology contained in a *nlc* topology?

THEOREM 5.1. *Let $(X, \mathcal{T})$ be an arbitrary Hausdorff topological space. There exists a topology $\mathcal{U}$ for X such that $\mathcal{T}\subseteq\mathcal{U}$, $\mathcal{U}$ is nlc iff $\phi\neq O\in\mathcal{T}$ implies that O is infinite.*

PROOF. If there existed a nonempty finite O in $\mathcal{T}$, then clearly $\mathcal{U}$ would not be *nlc* for any $\mathcal{U}\supseteq\mathcal{T}$.

Let us assume then that $\phi\neq O\in\mathcal{T}$ implies that O is infinite. Theorem 5.1 will follow from the following lemmas.

LEMMA 5.2. *Let $(X, \mathcal{T})$ be a space in which every nonempty open set is infinite. Then there exists a topology $\mathcal{U}^*$ for X such that* (1) $\mathcal{T}\subseteq\mathcal{U}^*$, (2) *all nonempty sets in $\mathcal{U}^*$ are infinite and* (3) *$\mathcal{U}^*$ is maximal relative to* (1) *and* (2).

The proof is an easy exercise using Zorn's lemma and will be omitted.

LEMMA 5.3. *Let $(X, \mathcal{T})$ be a space in which every nonempty open set is infinite and suppose $\mathcal{T}$ is maximal relative to this property. If $A\subseteq X$ and $O^*\subseteq A\subseteq c(O^*)$ for some $O^*\in\mathcal{T}$, then $A\in\mathcal{T}$.*

PROOF. Let $\mathcal{U}=sup\{\mathcal{T}, \{\phi, A, X\}\}$. It suffices to show that if $\phi\neq U\in\mathcal{U}$, then U is infinite.

Let $x\in U$. Clearly we may assume that $O^*\neq\phi$. If $x\in O\subseteq U$ for some $O\in\mathcal{T}$ or if $x\in A\subseteq U$, then U is infinite. Assume then that $x\in O\cap A\subseteq U$ for some $O\in\mathcal{T}$. Then $x\in O\cap c(O^*)$ and hence $O\cap O^*$ is nonempty. But $O\cap O^*\subseteq O\cap A\subseteq U$ and thus U is infinite.

LEMMA 5.4. *Let $(X, \mathcal{T})$ be an infinite Hausdorff space. There exist O_1, O_2, ..., O_n,.. in $\mathcal{T}$ such that $O_i\neq\phi$ for all i and $O_i\cap O_j=\phi$ when $i\neq j$.*

PROOF. See Theorem 5.2.3 in [2].

LEMMA 5.5. *Let* $(X, \mathscr{T})$ *be a space in which every nonempty open set is infinite and suppose* $\mathscr{T}$ *is maximal relative to this property. Then* $(X, \mathscr{T})$ *is extremally disconnected.*

PROOF. Let $O_1 \cap O_2 = \phi$, $O_i \in \mathscr{T}$. We assert that $c(O_1) \cap c(O_2) = \phi$. Suppose that $x \in c(O_1) \cap c(O_2)$. Then $O_1 \cup \{x\}$ and $O_2 \cup \{x\}$ are open by Lemma 5.3. It follows then that $\{x\}$ is in $\mathscr{T}$ and $\{x\}$ is finite, a contradiction.

LEMMA 5.6. *Let* $(X, \mathscr{T})$ *be a Hausdorff space in which every nonempty open set is infinite. Suppose* $\mathscr{T}$ *is maximal relative to this property. Then* $(X, \mathscr{T})$ *is nlc.*

PROOF. Deny. Then there exists an open set $O \in \mathscr{T}$ and a compact set K such that $\phi \neq O \subseteq K \subseteq X$. Since O is infinite and Hausdorff (as a subspace), there exist disjoint nonempty open set O_i for which $O_i \subseteq O$ for $i \geq 1$ by Lemma 5.4. Let $U = \cup \{O_i : i \geq 1\}$. Then $c(U) - U \subseteq K$ and $c(U) - U$ is compact. Then $\{x\} \cup U$ is open for all $x \in c(U) - U$ by Lemma 5.3. It follows then that $c(U) - U$ is finite. Let $U_1 = \cup \{O_{2i} : i \geq 1\}$ and $U_2 = \cup \{O_{2i-1} : i \geq 1\}$. Then $c(U) - U = (c(U_1) - U_1) \cup (c(U_2) - U_2)$ and $(c(U_1) - U_1) \cap (c(U_2) - U_2) \subseteq c(U_1) \cap c(U_2) = \phi$ by Lemma 5.5. Thus $c(U_1) - U_1$ or $c(U_2) - U_2$ has fewer elements than $c(U) - U$. Continuing in this way we get a sequence $O_{n_1}, O_{n_2}, \ldots$ for which $c(O_{n_1} \cup O_{n_2} \cup \ldots) = O_{n_1} \cup O_{n_2} \cup \ldots$ But $c(O_{n_1} \cup \ldots) \subseteq K$ and hence $c(O_{n_1} \cup \ldots)$ is compact. But $\{O_{n_i} : i \geq 1\}$ is an open cover of $c(O_{n_1} \cup \ldots)$ with no finite subcover, a contradiction.

Theorem 5.1 now follows from Lemma 5.2 and Lemma 5.6.

6. Maximal *nlc* topologies

THEOREM 6.1. *Let* $\mathscr{T}_\alpha$ *be a nlc topology for* X *for each* $\alpha \in \Delta$. *Suppose* $\mathscr{T}_\alpha \subseteq \mathscr{T}_\beta$ *or* $\mathscr{T}_\beta \subseteq \mathscr{T}_\alpha$ *for all* α, β *in* Δ. *If* $\mathscr{U} = sup\{\mathscr{T}_\alpha : \alpha \in \Delta\}$, *then* $\mathscr{U}$ *is a nlc topology for* X.

PROOF. Suppose $\phi \neq U \subseteq K \subseteq X$ where $U \in \mathscr{U}$ and K is compact relative to $\mathscr{U}$. There exists then an $O_\alpha \in \mathscr{T}_\alpha$ for some α for which $\phi \neq O_\alpha \subseteq U \subseteq K$. But K is $\mathscr{T}_\alpha$ compact and thus $(X, \mathscr{T}_\alpha)$ is not *nlc*, a contradiction.

COROLLARY 6.2. *Let* $(X, \mathscr{T})$ *be nlc. Then there exists a topology* $\mathscr{U}$ *for* X *such that* (1) $\mathscr{T} \subseteq \mathscr{U}$ (2) $\mathscr{U}$ *is nlc and* (3) $\mathscr{U}$ *is maximal relative to* (1) *and* (2).

PROOF. Use Zorn's lemma and Theorem 6.1.

COROLLARY 6.3. *Let $(X, \mathscr{T})$ be a Hausdorff space in which every nonempty set is infinite. There exists then a topology $\mathscr{U}$ for X such that* (1) $\mathscr{T}\subseteq\mathscr{U}$ (2) $\mathscr{U}$ *is nlc and* (3) $\mathscr{U}$ *is maximal relative to* (1) *and* (2).

PROOF. This follows from Corollary 6.2 and Theorem 5.1.

DEFINITION 6.3. We will call a topology $\mathscr{U}$ for X *maximal nowhere locally compact* (written henceforth as *mnlc*) if $\mathscr{U}$ is *nlc* and $\mathscr{U}\subseteq\mathscr{W}$ implies $\mathscr{U}=\mathscr{W}$ where $\mathscr{W}$ is *nlc*.

THEOREM 6.4. *Let $(X, \mathscr{T})$ be a space and $\mathscr{T}$ a mnlc topology. Then $(X, \mathscr{T})$ is a T_1-space.*

PROOF. This follows immediately from Theorem 4.1.

THEOREM 6.5. *Let $(X, \mathscr{T})$ be a space in which $\mathscr{T}$ is a mnlc topology. Then $\mathscr{T}$ contains all of its semi-open sets, that is, if $O^*\subseteq A\subseteq c(O^*)$ where $O^*\in\mathscr{T}$, then $A\in\mathscr{T}$.*

PROOF. This follows from Theorem 4.4.

EXAMPLE 6.6. The space of rationals with the usual topology is *nlc* but not *mnlc*. $\{r|0\leq r<1,\ r \text{ rational}\}$ is semi-open in the rationals, but not open.

THEOREM 6.7. *Let $(X, \mathscr{T})$ be a space in which $\mathscr{T}$ is mnlc. Then every compact set in X is closed.*

PROOF. This follows immediately from Theorem 4.2.

THEOREM 6.8. *Let $(X, \mathscr{T})$ be a space in which $\mathscr{T}$ is a mnlc topology. Then $\mathscr{T}$ contains all of the dense sets in X.*

PROOF. This follows from Theorem 6.7 and Theorem 4.3.

THEOREM 6.9. *Let $(X, \mathscr{T})$ be a space in which $\mathscr{T}$ is a mnlc topology. Then $(X, \mathscr{T})$ is extremally disconnected.*

PROOF. This follows immediately from Theorem 6.5.

THEOREM 6.10. *Let $(X, \mathscr{T})$ be a space in which $\mathscr{T}$ is a mnlc topology. Then all compact sets are finite.*

PROOF. Let $K\subseteq X$, K compact. Then K is closed and $\mathscr{C}c\mathscr{C}K=\phi$. Thus $\mathscr{C}K$ is open and dense in X. By Theorem 6.5, it follows that $\{x\}\cup\mathscr{C}K$ is open for all

x in X. But $\{\{x\}\cup\mathcal{C}K : x\in K\}$ is an open cover of K and it follows then that K is finite.

7. Additivity theorems

THEOREM 7.1. *Let $(X, \mathcal{T})$ be a space and $A_\alpha\subseteq X$ for all $\alpha\in\Delta$. If $X=\cup\{A_\alpha : \alpha\in\Delta\}$, A_α is closed for all α and $(A_\alpha, A_\alpha\cap\mathcal{T})$ is nlc, then $(X, \mathcal{T})$ is nlc.*

PROOF. Suppose $\phi\neq O\subseteq K\subseteq X$ where $O\in\mathcal{T}$ and K is compact. Then there exists an $\alpha\in\Delta$ for which $\phi\neq A_\alpha\cap O\subseteq A_\alpha\cap K\subseteq A_\alpha$. Then $A_\alpha\cap K$ is compact and $(A_\alpha, A_\alpha\cap\mathcal{T})$ is not *nlc*, a contradiction.

THEOREM 7.2. *Let $(X, \mathcal{T})$ be a space and $O_\alpha\in\mathcal{T}$ for all $\alpha\in\Delta$. Suppose $X=\cup\{O_\alpha : \alpha\in\Delta\}$ and $O_\alpha\subseteq O_\beta$ or $O_\beta\subseteq O_\alpha$ for all α, β in Δ. Then $(X, \mathcal{T})$ is nlc iff O_α is nlc for all $\alpha\in\Delta$.*

PROOF. If X is *nlc*, then O_α is *nlc* by Theorem 2.1. Assume that O_α is *nlc* for all $\alpha\in\Delta$. Suppose $\phi\neq O\subseteq K\subseteq X$ where K is compact and $O\in\mathcal{T}$. Then $K\subseteq O_\beta$ for some β and $\phi\neq O\subseteq K\subseteq O_\beta$. Then O_β is not *nlc* a contradiction.

EXAMPLE 7.3. A union of two open *nlc* sets need not be *nlc*. Let $(X, \mathcal{T})$ be the space of rationals with the usual topology. Let D be the diadic rationals and $E=X-D$. Let $a\neq b$, $a\notin X$, $b\notin X$. Let $Y=X\cup\{a, b\}$ and let $\mathcal{U}=\mathcal{T}\cup\{\{a\}\cup O : O\in\mathcal{T}$ and $O\supseteq D\}\cup\{\{b\}\cup O : O\in\mathcal{T}$ and $O\supseteq E\}$. Then clearly $\mathcal{U}$ is a topology for Y and $(Y, \mathcal{U})$ is compact and therefore not *nlc*. Let $U_1=\{a\}\cup X$ and $U_2=\{b\}\cup X$. U_1 and U_2 are open in $(Y, \mathcal{U})$. We now show that U_1 is *nlc*. Suppose $\phi\neq U^*\subseteq K\subseteq U_1$ where $U^*\in U_1\cap\mathcal{U}$ and K is compact.

CASE 1. $a\notin K$. Then $\phi\neq U^*\subseteq K\subseteq X$ and $U^*\in\mathcal{T}$, a contradiction.

CASE 2. $a\in K$. There exist rationals $r<s$ for which $(r, s)\cap X\subseteq U^*\subseteq K$. Let $z\in(r, s)$, z irrational. Select $l_1<l_2<\ldots$ in $(r, s)\cap E$ so that $\lim l_i=z$ in the space of reals. If $F_i=\{l_i, l_{i+1},\ldots\}$, then $\{F_i : i\geq 1\}$ is a family of closed sets in K with the finite intersection property. But $\cap\{F_i : i\geq 1\}=\phi$, a contradiction.

U_2 can be shown to be *nlc* by a similar argument.

LEMMA 7.4. *Let $(X, \mathcal{T})$ be a space and $X=A\cup B$. Suppose A and B are both nlc and $\mathcal{C}A\subseteq O_1$, $\mathcal{C}B\subseteq O_2$, $O_i\in\mathcal{T}$ and $O_1\cap O_2=\phi$. Then $(X, \mathcal{T})$ is nlc.*

PROOF. Suppose $\phi\neq O\subseteq K\subseteq X$ where $O\in\mathcal{T}$ and K is compact.

CASE 1. $O\cap O_1\neq\phi$. Then $\phi\neq O\cap O_1\subseteq K\cap\mathcal{C}O_2\subseteq B$. But $K\cap\mathcal{C}O_2$ is compact and thus B is not *nlc*, a contradiction.

CASE 2. $O \cap O_1 = \phi$. Then $\phi \neq O \subseteq K \cap \mathcal{C}O_1 \subseteq A$. But $K \cap \mathcal{C}O_1$ is compact and hence A is not *nlc*, a contradiction.

COROLLARY 7.5. *Let $(X, \mathcal{T})$ be a normal space and let $X = O_1 \cup O_2$ where O_1 and O_2 are in $\mathcal{T}$. If O_1 and O_2 are nlc, then X is nlc.*

PROOF. $\mathcal{C}O_1$ and $\mathcal{C}O_2$ are disjoint closed sets and hence can be separated by disjoint open sets.

THEOREM 7.6. *Let $(X, \mathcal{T})$ be a Hausdorff space and let $X = U \cup V$ where U and V are open and nlc. Then X is nlc.*

PROOF. Suppose $\phi \neq O \subseteq K \subseteq X$ where $O \in \mathcal{T}$ and K is compact. Now $\mathcal{C}U \cap \mathcal{C}V = \phi$ and thus $K \cap \mathcal{C}U$ and $K \cap \mathcal{C}V$ are disjoint compact sets in a Hausdorff space. There exist open sets O_1 and O_2 for which $K \cap \mathcal{C}V \subseteq O_1$ and $K \cap \mathcal{C}U \subseteq O_2$ and $O_1 \cap O_2 = \phi$. Now $O \cap O_1 \subseteq K \cap \mathcal{C}O_2 \subseteq K \cap U \subseteq U$ and $K \cap \mathcal{C}O_2$ is compact. Since U is *nlc*, it follows that $O \cap O_1 = \phi$. Thus $\phi \neq O \subseteq K \cap \mathcal{C}O_1 \subseteq V$ and V is not *nlc*, a contradiction.

COROLLARY 7.7. *Let $(X, \mathcal{T})$ be a Hausdorff space and let $X = O_1 \cup \ldots \cup O_n$ where $O_i \in \mathcal{T}$ and O_i is nlc for ecah i, then X is nlc.*

PROOF. Use the induction.

COROLLARY 7.8. *Let $(X, \mathcal{T})$ be a Hausdorff space and let $X = \cup \{O_\alpha : \alpha \in \Delta\}$ where $O_\alpha \in \mathcal{T}$ and O_α is nlc for all $\alpha \in \Delta$. Then $(X, \mathcal{T})$ is nlc.*

PROOF. If $\phi \neq O \subseteq K \subseteq X$ where $O \in \mathcal{T}$ and K is compact, then $\phi \neq O \subseteq K \subseteq O_{\alpha_1} \cup \ldots \cup O_{\alpha_n}$ and $O_{\alpha_1} \cup \ldots \cup O_{\alpha_n}$ is not *nlc*, a contradiction.

THEOREM 7.9. *Let $(X, \mathcal{T})$ be a regular space (Hausdorff not assumed) and suppose $X = O \cup A$ where $O \in \mathcal{T}$ and O and A are nlc. Then $(X, \mathcal{T})$ is nlc.*

PROOF. Suppose $\phi \neq O^* \subseteq K \subseteq X$ where O^* is open and K is compact.

CASE 1. $O \cap O^* \neq \phi$. Since $(X, \mathcal{T})$ is regular, there exists an $O^\# \in \mathcal{T}$ such that $\phi \neq O^\# \subseteq c(O^\#) \subseteq O \cap O^* \subseteq O^* \subseteq K$. Thus $c(O^\#)$ is compact and $\phi \neq O^\# \subseteq c(O^\#) \subseteq O$ and O is not *nlc*, a contradiction.

CASE 2. $O \cap O^* = \phi$. Then $\phi \neq O^* \subseteq K \cap \mathcal{C}O \subseteq A$. But $K \cap \mathcal{C}O$ is compact and hence A is not *nlc*, a contradiction.

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