Kyungpook Math. J.
Volume 22, Number 2
December, 1982

EXTENSIONS OF TOPOLOGICAL ORDERED SPACES

By T.H. Choe and S.S. Hong*

0. Introduction

Since Nachbin has shown in [10] that every completely regular ordered space has an ordered compactification, now known as the Nachbin compactification, recently there has been a growing interest in extensions of topological ordered spaces (see [2], [3], [4], [6], [8], [10], [11], [12], [13]).

Using filter traces, Banaschewski has characterized in [1] various extensions of topological spaces. In particular, he has shown that for any topological space there are two extremal extensions namely the strict one and simple one and that any extension of a topological space is in a sense situated between the strict extension and the simple extension.

In [12], Y.S. Park has introduced the concept of bifilters with which he has constructed the Wallman type compactification of a topological ordered space. Also in [6], Hong has constructed the zero-dimensional ordered compactification with maximal clopen bifilters.

Our purpose to write this paper is to set up a systematic method to study extensions of topological ordered spaces as Banaschewski does for those of topological spaces in [1]. In this direction, we use open bifilters on Hausdorff convex ordered spaces to characterize their extensions.

It is shown that the underlying ordered set of a Hausdorff convex extension of a topological ordered space is completely recovered by the trace open bifilters. Then considering any family of open bifilters extending neighborhood bifilters, we construct the strict and simple extensions of a Hausdorff convex ordered space which have the same properties as those of topological spaces. As a byproduct, we have a characterization of o-H-closed ordered spaces by the fact every maximal open bifilter is convergent. Also, we show that every Hausdorff convex ordered space has an o-H-closed extension.

For the terminology not introduced in this paper, we refer to [10].

*This research was aided by MOE Research Grant (1981).

1. Extensions of convex topological ordered spaces

In this section, we show that the order on any Hausdorff convex extension of a topological ordered space X can be recovered by a natural order on a set of open bifilters on X. Thus open bifilters are shown to be a nice tool for the study of extensions of topological ordered spaces as open filters are for that of extensions of topological spaces.

1.1. DEFINITION. A topological ordered space $(X, \mathscr{T}, \leqq)$ is said to be *convex* if the set of all open increasing sets and all open decreasing sets forms a subbase for $\mathscr{T}$, equivalently, for any neighborhood N of $x \in X$, there is an increasing open neighborhood U of x and a decreasing open neighborhood V of x with $U \cap V \subseteq N$.

Let *TPOS* denote the category of topological ordered spaces and continuous isotones and *CTPOS* the full subcategory of *TPOS* formed by convex topological ordered spaces.

1.2. PROPOSITION. *The category CTPOS is closed under the initial mono-sources in TPOS.*

PROOF. Observe that a mono-source $(f_i)_I$ in *TPOS* is initial iff it is initial in *Top* and in the category *Ord* of ordered sets and isotones. Then the statement is immediate from the fact that the inverse image of an increasing (decreasing, resp.) set under an isotone is again increasing (decreasing, resp.). We omit the detail of the proof.

Using the above proposition and the fact that *TPOS* is an (epi, initial mono-sources) category, the following is immediate from [5].

1.3. COROLLARY. *CTPOS is epireflective in TPOS and hence CTPOS is productive and hereditary in TPOS.*

We note that real line with the usual topology and the usual order is convex and hence every completely regular ordered space is convex. Thus every compact ordered space is convex as known in [10].

The following definition is due to Park [12].

1.4. DEFINITION. A pair $(\mathscr{F}, \mathscr{G})$ of filters on an ordered set X is said to be a *bifilter* on X if it satisfies the following:

(1) $\mathscr{F}$ has a base consisting of increasing sets,

(2) $\mathscr{G}$ has a base consisting of decreasing sets,

(3) For any $F\in\mathscr{F}$ and any $G\in\mathscr{G}$, $F\cap G\neq\phi$.

From (3) of the above definition, for any bifilter $(\mathscr{F}, \mathscr{G})$, the join filter $(\mathscr{F}\vee\mathscr{G})$ of $\mathscr{F}$ and $\mathscr{G}$ exists.

1.5. DEFINITION. Let X be a topological ordered space.

(1) A bifilter $(\mathscr{F}, \mathscr{G})$ on X is said to be an *open bifilter* if $\mathscr{F}$ and $\mathscr{G}$ have bases consisting of open increasing sets and open decreasing sets respectively.

(2) A bifilter $(\mathscr{F}, \mathscr{G})$ on X is said to *converge* to x if $\mathscr{F}\vee\mathscr{G}$ converges to x in X. In this case, x is also said to be a *limit* of the bifilter $(\mathscr{F}, \mathscr{G})$.

NOTATION. Let X be a topological ordered space and for any $x\in X$, let $\vartheta_X(x)$ ($\mathscr{D}_X(x)$, resp.) denote the filters generated by increasing (decreasing, resp.) open neighborhoods of x. If there is no confusion on X, then $\vartheta_X(x)$ ($\mathscr{D}_X(x)$, resp,) will be denoted by $\vartheta(x)$($\mathscr{D}(x)$, resp.). Clearly $(\vartheta(x), \mathscr{D}(x))$ is an open bifilter. If X is a convex topological ordered space, then a bifilter $(\mathscr{F}, \mathscr{G})$ converges to x if $\vartheta(x)\subseteq\mathscr{F}$ and $\mathscr{D}(x)\subseteq\mathscr{G}$.

1.6. DEFINITION. Let X and T be topological ordered spaces. Then T is said to be an *extension* of X if X is iseomorphic i.e., homeomorphic and order isomorphic, with a dense subspace of T.

Let T be a Hausdorff convex extension of X, i.e., T is a Hausdorff convex ordered space of which X is a dense subspace. For any $t\in T$, let $\mathscr{F}_t$ be the trace filter of $\vartheta_T(t)$ on X and $\mathscr{G}_t$ the trace filter of $\mathscr{D}_T(t)$ on X. Since X is a dense subspace of T, $(\mathscr{F}_t, \mathscr{G}_t)$ is an open bifilter on X and for any $t\in X$, $\mathscr{F}_t=\vartheta_X(t)$ and $\mathscr{G}_t=\mathscr{D}_X(t)$. Since T is a Hausdorff space, for any $t\in T-X$, $(\mathscr{F}_t, \mathscr{G}_t)$ is not convergent in X.
In all, $\Phi(T)=\{(\mathscr{F}_t, \mathscr{G}_t)\mid t\in T\}$ is a set of open bifilters on X extending $\{(\vartheta_X(x), \mathscr{D}_X(x))\mid x\in X\}$ such that for $t\in T-X$, $(\mathscr{F}_t, \mathscr{G}_t)$ is not convergent.

Now we introduce the order relation on $\Phi(T)$ as follows:

$(\mathscr{F}_t, \mathscr{G}_t)\leqq(\mathscr{F}_{t'}, \mathscr{G}_{t'})$ iff $\mathscr{F}_t\subseteq\mathscr{F}_{t'}$ and $\mathscr{G}_{t'}\subseteq\mathscr{G}_t$.

Using this notion, one has the following:

1.7. THEOREM. *For any Hausdorff convex extension T of X, $(\Phi(T), \leqq)$ is an ordered set which is order isomorphic with T.*

PROOF. The first part is immediate from the fact that the order relation on $\Phi(T)$ is defined by the inclusion relation of filters. For the second part, define a map $e: T \longrightarrow \Phi(T)$ by $e(t)=(\mathscr{F}_t, \mathscr{G}_t)$. Clearly e is onto. Suppose $t \neq t'$ and hence we may assume $t \not\leq t'$. Since the order in a Hausdorff convex ordered space is continuous, there is an increasing open neighborhood U of t and a decreasing open neighborhood V of t' with $U \cap V = \phi$. Hence $U \cap X \in \mathscr{F}_t$ and $V \cap X \in \mathscr{G}_{t'}$. Thus $(\mathscr{F}_t, \mathscr{G}_t) \neq (\mathscr{F}_{t'}, \mathscr{G}_{t'})$, i.e,. $e(t) \neq e(t')$. Hence e is one-one onto.

If $t \leq t'$ in T, then $\vartheta_T(t) \subseteq \vartheta_T(t')$ and $\mathscr{D}_T(t') \subseteq \mathscr{D}_T(t)$, so that $\mathscr{F}_t \subseteq \mathscr{F}_{t'}$ and $\mathscr{G}_{t'} \subseteq \mathscr{G}_t$. Hence $e(t) \leq e(t')$. Thus e is an isotone. Suppose $t \not\leq t'$ in T. Again using the fact that the order in T is continuous, there is an increasing open neighborhood U of t and a decreasing open neighborhood V of t' with $U \cap V = \phi$. Hence $U \cap X \in \mathscr{F}_t$ and $V \cap X \in \mathscr{G}_{t'}$. If $\mathscr{F}_t \subseteq \mathscr{F}_{t'}$, then $U \cap X \in \mathscr{F}_{t'}$ which is a contradiction to $V \cap X \in \mathscr{G}_{t'}$. Thus $\mathscr{F}_t \not\subseteq \mathscr{F}_{t'}$, i.e., $e(t) \not\leq e(t')$. In all e is an order isomorphism.

The above theorem amounts to saying that the underlying ordered set of a Hausdorff convex extension of X is completely determined by trace open bifilters on X.

2. Strict extensions of convex topological ordered spaces

In this section we introduce strict extensions of convex topological ordered spaces and investigate their properties.

For a Hausdorff convex ordered space X, let T be a super set of X, i.e., $X \subseteq T$. Consider a family $\{(\mathscr{F}_t, \mathscr{G}_t) | t \in T\}$ of open bifilters on X such that for any $x \in X$, $(\mathscr{F}_t, \mathscr{G}_t) = (\vartheta(x), \mathscr{D}(x))$ and for any $t \in T - X$, $(\mathscr{F}_t, \mathscr{G}_t)$ is not convergent. For any increasing open set U of X, let $\hat{U} = \{t \in T | U \in \mathscr{F}_t\}$ and for any decreasing open set V of X, let $\hat{V} = \{t \in T | V \in \mathscr{G}_t\}$.

Using the above notation, one has the following.

2.1. PROPOSITION. *Let U, V be increasing open sets in X. Then one has,*

(1) $\hat{U} = \phi$ *iff* $U = \phi$.

(2) $\hat{U} \cap \hat{V} = (U \cap V)^\wedge$

(3) $\hat{U} \cap X = U$.

Dually the corresponding properties for decreasing open sets in X also hold.

PROOF. It is straightforward and we left the proof to the readers.

Now we are ready to introduce the order and topology on T. Let $\mathscr{T}$ be the topology on T generated by $\{\hat{U}|U$: increasing open set in $X\} \cup \{\hat{V}|V$: decreasing open set in $X\}$. Using the result in Section 1, we define the order on T as follows: $t \leqq t'$ iff $\mathscr{F}_t \subseteq \mathscr{F}_{t'}$ and $\mathscr{G}_{t'} \subseteq \mathscr{G}_t$. Then $(T, \mathscr{T}, \leqq)$ is a topological ordered space which will be again denoted by T.

2.2. THEOREM. *The topological ordered space T is a convex extension of X.*

PROOF. Since X is a Hausdorff convex ordered space, the inclusion map $X \longrightarrow T$ is an order isomorphism. Using (3) and the dual of (3) in Proposition 2.1, X is a topological subspace of T. For any $t \in T$, take an increasing open set U and a decreasing open set V with $t \in \hat{U} \cap \hat{V}$, then $U \in \mathscr{F}_t$ and $V \in \mathscr{G}_t$. Hence $U \cap V \in \mathscr{F}_t \vee \mathscr{G}_t$. Since $U \cap V \subseteq \hat{U} \cap \hat{V}$, $\mathscr{F}_t \vee \mathscr{G}_t$ converges to t and $X \in \mathscr{F}_t \vee \mathscr{G}_t$. Thus $t \in \bar{X}$. Therefore T is an extension of X. For any increasing open set U in X and $t \in \hat{U}$, suppose $t \leqq t'$. Then $U \in \mathscr{F}_t \subseteq \mathscr{F}_{t'}$ so that $t' \in \hat{U}$. Thus $\hat{U}$ is again increasing in T and dually for decreasing open set V in X, $\hat{V}$ is also decreasing in T. Hence T is a convex topological ordered space.

2.3. DEFINITION. The extension T of a Hausdorff convex ordered space X constructed in Theorem 2.2 will be called the *strict extension of X associated with the set* $\{(\mathscr{F}_t, \mathscr{G}_t)|t \in T\}$ of open bifilters on X. Also an extension E of a topological ordered space X is said to be *strict* if E is a strict extension of X associated with some set of open bifilters on X.

Let us try to find a condition for which the strict extension T of X would be a Hausdorff extension.

2.4. DEFINITION. (1) For open bifilters $(\mathscr{F}, \mathscr{G})$ and $(\mathscr{F}', \mathscr{G}')$ on a topological ordered space X, $(\mathscr{F}, \mathscr{G})$ is said to be *contained* in $(\mathscr{F}', \mathscr{G}')$ if $\mathscr{F} \subseteq \mathscr{F}'$ and $\mathscr{G} \subseteq \mathscr{G}'$. In case we write $(\mathscr{F}, \mathscr{G}) \subseteq (\mathscr{F}', \mathscr{G}')$.

(2) An open bifilter is said to be a *maximal open bifilter* if it is a maximal element with respect to the above relation $\subseteq$.

By Zorn's lemma, for any open bifilter $(\mathscr{F}, \mathscr{G})$, there is a maximal open bifilter $(\mathscr{H}, \mathscr{K})$ with $(\mathscr{F}, \mathscr{G}) \subseteq (\mathscr{H}, \mathscr{K})$. Using this, one can easily show that an open bifilter $(\mathscr{H}, \mathscr{K})$ is maximal iff for any increasing open set U with $U \cap H \cap K \neq \phi$ for all $H \in \mathscr{H}$, $K \in \mathscr{K}$, U belongs to $\mathscr{H}$ and dually for decreasing open set V with $V \cap H \cap K \neq \phi$ for all $H \in \mathscr{H}$, $K \in \mathscr{K}$, V belongs to $\mathscr{K}$.

With the same notation as that in Theorem 2.1, one has,

2.5. THEOREM. *Suppose X is a Hausdorff convex ordered space. If for each $t \in$*

$T-X$, $(\mathscr{G}_t, \mathscr{G}_t)$ is a maximal open bifilter, then T is a Hausdorff convex extension of X.

PROOF. Take two distinct points s and t in T. If $s, t \in X$, then either $s \not\leq t$ or $t \not\leq s$. Assume $s \not\leq t$. Then there exists an increasing open neighborhood U of t in X and a decreasing open neighborhood V of s in X with $U \cap V = \phi$. Then $\hat{U}$ and $\hat{V}$ are clearly disjoint neighborhoods of s and t in T respectively. If $s, t \in T-X$, then $(\mathscr{F}_s, \mathscr{G}_s)$ and $(\mathscr{F}_t, \mathscr{G}_t)$ are distinct maximal open bifilters. Hence either $\mathscr{F}_s \not\subseteq \mathscr{F}_t$ or $\mathscr{G}_s \not\subseteq \mathscr{F}_t$. Suppose $\mathscr{F}_s \not\subseteq \mathscr{F}_t$, then there is $U \in \mathscr{F}_s - \mathscr{F}_t$. By the above characterization of maximal open bifilters, there is $H \in \mathscr{F}_t$ and $K \in \mathscr{G}_t$ such that $U \cap H \cap K = \phi$. Then $\hat{U}$ and $\hat{H} \cap \hat{K}$ are again disjoint neighborhoods of t and s in T respectively. If $\mathscr{G}_s \not\subseteq \mathscr{G}_t$, one can also find disjoint neighborhoods of s and t by the same argument.

Finally suppose $s \in T-X$ and $t \in X$, then $(\mathscr{F}_s, \mathscr{G}_s)$ can not converge to t. Thus one has either $\mathscr{F}_t = \mathscr{I}_X(t) \not\subseteq \mathscr{F}_s$ or $\mathscr{G}_t = \mathscr{D}_X(t) \not\subseteq \mathscr{G}_s$. Assume $\mathscr{I}_X(t) \not\subseteq \mathscr{F}_s$, then there is $U \in \mathscr{I}_X(t) - \mathscr{F}_s$. Thus there is $H \in \mathscr{F}_s$ and $K \in \mathscr{G}_s$ such that $U \cap H \cap K = \phi$. Therefore $\hat{U}$ and $\hat{H} \cap \hat{K}$ are disjoint neighborhoods of t and s in T respectively. Similarly, for the case of $\mathscr{D}_X(t) \not\subseteq \mathscr{G}_s$, one can find disjoint neighborhoods of s and t. This completes the proof.

Using the above, we have the following interesting result.

2.6. DEFINITION. A Hausdorff convex ordered space X is said to be *o-H-closed* if whenever X is a subspace of any Hausdorff convex ordered space T, X is closed in T.

If the order in a topological ordered space X is discrete, then X is o-H-closed iff it is H-closed. Moreover, it is clear that every compact ordered space is o-H-closed.

2.7. THEOREM. *Let X be a Hausdorff convex ordered space. Then X is o-H-closed iff every maximal open bifilter on X is convergent.*

PROOF. Suppose X is o-H-closed and there is a maximal open bifilter $(\mathscr{U}, \mathscr{K})$ in X which is not convergent. Take any element $w \notin X$ and let $T = X \cup \{w\}$. We denote the strict extension of X associated with $\{(\mathscr{I}(x), \mathscr{D}(x)) \mid x \in X\} \cup \{(\mathscr{U}, \mathscr{K})\}$ by T, i.e. $(\mathscr{F}_w, \mathscr{G}_w) = (\mathscr{U}, \mathscr{K})$. Since $(\mathscr{U}, \mathscr{K})$ is a maximal open bifilter, by the above theorem T is a Hausdorff extension of X, which is a contradiction to the fact that X is o-H-closed. Conversely, suppose X is a subspace of a Hausdorff convex ordered space T. Pick any $t \in \overline{X}$ and let $\mathscr{L}(\mathcal{L}$, resp.) be the trace filter of $\vartheta_T(t)(\mathscr{D}_T(t)$, resp.) on X. Since $(\mathscr{L}, \mathcal{L})$ is an open bifilter on X, there is a maximal open bifilter $(\mathscr{F}, \mathscr{G})$ on X with $(\mathscr{L}, \mathcal{L}) \subseteq (\mathscr{F}, \mathscr{G})$. By the assumption, $(\mathscr{F}, \mathscr{G})$ converges to x in X and hence to x in T. Since $(\mathscr{L}, \mathcal{L})$ converges to t in T, $(\mathscr{F}, \mathscr{G})$ also converges to t. Since T is a Hausdorff space, $t = x \in X$. Hence X is a closed subspace of T.

For any Hausdorff convex extension T of X, let sT be the strict extension of X associated with $\Phi(T)$ as in Section 1. Then one has the following:

2.8. THEOREM. *For any Hausdorff convex extension T of X, the identity map $1_T : T \longrightarrow sT$ is a continuous order isomorphism.*

PROOF. By the definition of sT and $\Phi(T)$, it is clear that the map $\Phi(T) \longrightarrow sT$ defined by $(\mathscr{F}_t, \mathscr{G}_t) \longrightarrow t$, is an order isomorphism. Hence by Theorem 1.7, the identity map $1_T : T \longrightarrow sT$ is an order isomorphism. For any $t \in sT$, let U (V, resp.) be an increasing (decreasing, resp.) open set with $t \in \hat{U} \cap \hat{V}$. Hence $U \in \mathscr{F}_t = \vartheta(t)_X$ and $V \in \mathscr{G}_t = \mathscr{D}(t)_X$. Thus there is $U_1 \in \vartheta(t)$ and $V_1 \in \mathscr{D}(t)$ such that $U_1 \cap X \subseteq U$ and $V_1 \cap X \subseteq V$. Hence $U_1 \cap V_1 \subseteq \hat{U} \cap \hat{V}$. Thus $\hat{U} \cup \hat{V}$ is a neighborhood of t in T. This completes the proof.

2.9. DEFINITION. A convex ordered space X is said to be *regular* if for any $x \in X$ and any increasing neighborhood U of x there is an increasing open neighborhood N of x with $\overline{N} \subseteq U$, and dually for any decreasing neighborhood V of x, there is a decreasing open neighborhood M of x with $\overline{M} \subseteq V$.

Clearly, every completely regular ordered space is regular, and in particular, every compact ordered space is regular. The following theorem says that there are plenty of strict extensions.

2.10. THEOREM. *Every regular extension T of a topological ordered space X is a strict extension of X.*

PROOF. Let sT be the strict extension of X associated with $\Phi(T)$. By the above theorem, it is enough to show that the identity map $1_T : sT \longrightarrow T$ is continuous. Take any $t \in T$ and let $U(V$, resp.) be an increasing (decreasing, resp.) open neighborhood of t in T. Since T is regular, there is an increasing (decreasing, resp.) open neighborhood $U_1(V_1$, resp.) of t with $\overline{U}_1 \subseteq U(\overline{V}_1 \subseteq V$, resp.). Let $U_2 = U_1 \cap X$ and $V_2 = V_1 \cap X$. Then $\hat{U}_2 \cap \hat{V}_2$ is a neighborhood of t in sT. Take any $s\in\hat{U}_2$. For any increasing open neighborhood P of s in T and any decreasing open neighborhood Q of s in T, $P\cap X\in\mathscr{F}_s$ and $Q\cap X\in\mathscr{G}_s$. Since $U_2\in\mathscr{F}_s$, $U_2\cap(P\cap X)\cap(Q\cap X)\neq\phi$ and hence $U_1\cap(P\cap Q)\neq\phi$. Thus $s\in\overline{U}_1$, i.e., $\hat{U}_2\subseteq\overline{U}_1$. Dually one has $\hat{V}_2\subseteq\overline{V}_1$. Therefore $\hat{U}_2\cap\hat{V}_2\subseteq\overline{U}_1\cup\overline{V}_1\subseteq U\cap V$, so that $U\cap V$ is also a neighborhood of t in sT. This completes the proof.

3. Simple extensions

For a Hausdorff convex ordered space X, let $\{(\mathscr{F}_t,\mathscr{G}_t)|t\in T\}$ be a family of open bifilters on X and the order $\leqq$ on T as those in Section 2. Now let us define another topology $\mathscr{T}_p$ on T as follows: $\mathscr{T}_p=\{A\subseteq T|$ for any $t\in A$, there is $U\in\mathscr{F}_t$ and $V\in\mathscr{G}_t$ such that $(U\cap V)\cup\{t\}\subseteq A\}$. Then it is immediate that $\mathscr{T}_p$ is a topology on T. Hence $(T,\mathscr{T}_p,\leqq)$ is a topological ordered space which is denoted by T_p. Using the above notion, one has,

3.1. THEOREM. *The topological ordered space T_p is an extension of X.*

PROOF. By the proof of Theorem 2.2, X is an ordered subspace of T_p. Since any increasing (decreasing, resp.) open set in X is again open in T_p and any open set in T_p contained in X is also open in X, X is an open subspace of T_p. Moreover for any $t\in T_p$ and any open neighborhood A of t, there is $U\in\mathscr{F}_t$ and $V\in\mathscr{G}_t$ such that $(U\cap V)\cup\{t\}\subseteq A$. Hence $\phi\neq U\cap V\subseteq A\cap X$ so that $t\in\overline{X}$. Thus T_p is an extension of X.

3.2. DEFINITION. The extension T_p of X constructed in Theorem 3.1 will be called a *simple extension of X associated with the famly* $\{(\mathscr{F}_t,\mathscr{G}_t)|t\in T\}$ of open bifilters on X. An extension of X is also said to be *simple* if it is a simple extension of X associated with some family of open bifilters on X.

For any Hausdorff convex extension T of X, let pT be the simple extension of X associated with $\Phi(T)$.

.3. THEOREM. *For any Hausdorff convex extension T of X, the identity map $1_T: pT\longrightarrow T$ is a continuous order isomorphism.*

PROOF. As the proof of Theorem 2.8, 1_T is an order isomorphism. Take any open set A in T and let $t\in A$, then there is an increasing open neighborhood U of t and a decreasing open neighborhood V of t such that $U\cap V\subseteq A$. Then $U\cap X\in\mathscr{F}_t$, $V\cap X\in\mathscr{G}_t$ and $((U\cap X)\cup(V\cap X))\cup\{t\}\subseteq U\cup V\subseteq A$. Hence A is also open in pT. Thus 1_T is continuous.

Theorem 2.8 and 3.3 amount to saying that the topology on any Hausdorff convex extension T of X is coarser than that of the simple extension of X associated with $\Phi(T)$ and finer than that of the strict extension of X associated with $\Phi(T)$.

Using the same notation as that in Theorem 3.1, one has,

3.4. COROLLARY. *Suppose X is a Hausdorff convex ordered space. If for each $t \in T-X$, $(\mathscr{F}_t, \mathscr{G}_t)$ is a maximal open bifilter, then T_p is a Hausdorff extension of X.*

PROOF. By Theorem 2.5, the strict extension T of X associated with $\{(\mathscr{F}_t, \mathscr{G}_t) | t \in T\}$ is a Hausdorff convex extension of X. Since $\Phi(T) = \{(\mathscr{F}_t, \mathscr{G}_t) | t \in T\}$, $T_p = pT$. Since the identity map $1_T : T_p \longrightarrow T$ is continuous, T_p is also a Hausdorff extension of X.

We note that the simple extension T_p of X need not be convex. We define another order $\leqq'$ on T as follows: we keep the order on X and for any $t \in T-X$, t is not comparable with any element of T except t itself. Then clearly $(T, \mathscr{T}_p, \leqq')$ is a convex extension of X, which will be denoted by T'_p.

Using this T'_p, one has the following.

3.5. THEOREM. *Every Hausdorff convex ordered space has an o-H-closed extension.*

PROOF. Let X be a Hausdorff convex ordered space and T a set containing X such that $\{(\mathscr{F}_t, \mathscr{G}_t) | t \in T-X\}$ is the family of all non-convergent maximal open bifilters on X. Let $T'_p = (T, \mathscr{T}_p, \leqq')$ be the extension of X constructed as above. Then T'_p is a Hausdorff convex ordered space. By Theorem 2.7, it remains to show that every maximal open bifilter on T'_p is convergent. Let $(\mathscr{U}, \mathscr{V})$ be a maximal open bifilter on T'_p and let $\mathscr{L}(\mathscr{K}$, resp.) be the trace filter of $\mathscr{U}(\mathscr{V}$, resp.) on X. Then there is a maximal open bifilter $(\mathscr{F}, \mathscr{G})$ containing $(\mathscr{L}, \mathscr{K})$. If $(\mathscr{F}, \mathscr{G})$ converges to $x \in X$, then $(\mathscr{U}, \mathscr{V})$ also converges to x in T'_p. If $(\mathscr{F}, \mathscr{G})$ is not convergent, then there is some $t \in T$ with $(\mathscr{F}, \mathscr{G}) = (\mathscr{F}_t, \mathscr{G}_t)$. It is again easy to show that $(\mathscr{U}, \mathscr{V})$ converges to t in T'_p, and we omit the detail of the proof. This completes the proof.

REMARK. If a Hausdorff space X is considered as a topological ordered space with the discrete order, then the extension T'_p of X in Theorem 3.5 is precisely the Katětov extension of X.

McMaster University, Hamilton, Canada
Sogang University, Seoul, Korea

REFERENCES

[1] B. Banaschewski, *Extensions of topological spaces*, Can. Math. Bull. 7(1964), 1—22.

[2] T.H. Choe and D.C. Garcia, *Epireflective subcategories of partially ordered topological spaces*, Kyungpook Math. J. 13(1973), 97—107.

[3] T.H. Choe and Y.H. Hong, *Extensions of completely regular ordered spaces*, Pacific J. Math. 66(1976), 37—48.

[4] T.H. Choe and Y.S. Park, *Wallman's type order compactifications*, Pacific J. Math. 82(1979), 339—347.

[5] H. Herrlich, *Topological functors*, Gen. Top. Appl. 4(1974), 125—142.

[6] S.S. Hong, *0-dimensional compact ordered spaces*, Kyungpook Math. J. 20(1980), 158—167.

[7] S.S. Hong and L.D. Nel, *Spectral dualities involving mixed structures*, Categorical Aspects of Top and Analysis, Lect. Notes in Math. vol. 915, Springer (1982), 198—204.

[8] Y.H. Hong, *Studies on categories of universal topological algebras*, Doctoral Diss., McMaster Univ., Hamilton, 1974.

[9] M. Katětov, *Über H-abgeschlossene und bikompakte Räume*, Časopis Pěst. Mat. 69 (1940), 36—49.

[10] L. Nachbin, *Topology and order*, Van Nostrand, New York, 1965.

[11] B.H. Park, *On Wallman type extensions*, Kyungpook Math. J. 19(1979), 183—200.

[12] Y.S. Park, *Ordered topological spaces and topological semilattices*, Doctoral Diss., McMaster Univ., Hamilton, 1977.

[13] Y.S. Park, *Wallman's type order compactifications* II, Kyungpook Math. J. 19(1979), 151—158.