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ON A TWO-DIMENSIONAL H-H INTEGRAL TRANSFORM (II)

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1. Introduction

The two-dimensional H-H integral transform, introduced by the author [1] is defined and represented by the following integral equation:

$$\phi(p,q)=H-H\{f:p,q\}$$

$$=pq\int_0^\infty\int_0^\infty H_{s,t}^{r,0}[apx]\ H_{S,T}^{R,0}[bqy]\ H_1[mpx,\ nqy]\ f(x,y)dx\ dy$$

$$=pq\int_0^\infty\int_0^\infty H_{s,t}^{r,0}\left[apx\left|{}^{(k_j,\mu_j)_{1,s}}_{(u_j,\nu_j)_{1,t}}\right.\right]H_{S,T}^{R,0}\left[bqy\left|{}^{(g_j,G_j)_{1,S}}_{(h_j,H_j)_{1,T}}\right.\right]$$

$$\cdot H_{p_1,q_1:p_2,q_2;p_3,q_3}^{0,0:m_2,n_2;m_3,n_3}\left[{}^{\mathrm{mpx}}_{\mathrm{nqy}}\left|{}^{(a_j;\alpha_j,A_j)_{1,p_1}:(c_j,\varepsilon_j)_{1,p_2};(e_j,E_j)_{1,p_3}}_{(b_j;\beta_j,B_j)_{1,q_1}:(d_j,\delta_j)_{1,p_2};(f_j,F_j)_{1,q_3}}\right.\right]$$

$$\cdot f(x,y)\ dx\ dy\cdots\cdots(1.1)$$

provided that the double integral in the right-hand side of (1.1) exists. In (1.1)

$$H_{p,q}^{m,n}\left[x\left|{}^{(a_j,A_j)_{1,p}}_{(b_j,B_j)_{1,q}}\right.\right]$$

denotes the well-known Fox's H-function ([12]; see also, [9]), and $H_1[x,y]$ represents the H-function of two variables (with $n_1=0$) introduced by Mittal and Gupta ([13], p.117, see also, [4]). {For explanations of the notations employed and various symbols used here, the reader can refer the recent work by Goyal and Vasistha ([7], p.145).}

EXISTENCE CONDITIONS OF (1.1)

$$\text{(i)}\quad \left.\begin{aligned}S_1&=\sum_{j=1}^{r}\nu_j-\sum_{j=r+1}^{t}\nu_j-\sum_{j=1}^{s}\mu_j>0,\\ S_2&=\sum_{j=1}^{R}H_j-\sum_{j=R+1}^{T}H_j-\sum_{j=1}^{S}G_j>0,\end{aligned}\right\}\cdots\cdots(1.2)$$

$$|arg\ ap|<\frac{1}{2}S_1\pi\ and\ |arg\ bq|<\frac{1}{2}S_2\pi\cdots\cdots(1.3)$$

$$\text{(ii)}\quad \Delta_1=-\sum_{j=1}^{p_1}\alpha_j-\sum_{j=1}^{q_1}\beta_j+\sum_{j=1}^{m_2}\delta_j-\sum_{j=m_2+1}^{q_2}\delta_j+\sum_{j=1}^{n_2}\varepsilon_j-\sum_{j=n_2+1}^{p_2}\epsilon_j>0,$$

$$A_2 = -\sum_{j=1}^{p_1} A_j - \sum_{j=1}^{q_1} B_j + \sum_{j=1}^{m_3} F_j - \sum_{j=m_3+1}^{q_3} F_j + \sum_{j=1}^{n_3} E_j - \sum_{j=n_3+1}^{p_3} E_j > 0. \Big\} \cdots(1.4)$$

$$|arg\ mp| < \frac{1}{2} A_1 \pi \text{ and } |arg\ nq| < \frac{1}{2} A_2 \pi. \cdots\cdots(1.5)$$

(iii) *The existence conditions corresponding appropriately to the ones mentioned in the book by Mathai and Saxena* ([12], *p.*3, *Eqs.* (1.1.7) *and* (1.1.8)), *are satisfied by the both H-functions of one variable involved in* (1.1).

(iv) *The existence conditions, for the H-function of two variables given by Mittal and Gupta* ([13], *p.*119, (i) *and* (ii)), *are satisfied by the* $H_1\ [mpx,\ nqy]$ *occurring in* (1.1).

(v) $f(x, y)$ *is a real or complex valued function of two variables x and y defined on the region* $R: 0 \leqq x < \infty,\ 0 \leqq y < \infty$ *and such that the product* $x^{\alpha+\alpha'} y^{\beta+\beta'}$ $|f(x,y)|$ *is integrable over the finite region* $R\ (R_1, R_2): 0 \leqq x \leqq R_1:\ 0 \leqq y \leqq R_2$, $R_1 > 0,\ R_2 > 0$,

where

$$\alpha = \min_{1 \leqq j \leqq r} \{Re(u_j/\nu_j)\},\ \alpha' = \min_{1 \leqq j \leqq m_2} \{Re(d_j/\delta_j)\} \cdots\cdots(1.6)$$

$$\beta = \min_{1 \leqq j \leqq R} \{Re(h_j/H_j)\},\ \beta' = \min_{1 \leqq j \leqq m_3} \{Re(f_j/F_j)\} \cdots\cdots(1.7)$$

and

(vi) *the limit of the finite form of the double integral in* (1.1),

$$\text{with } \int_0^\infty\!\!\int_0^\infty \text{ replaced by } \int_0^{R_1}\!\!\int_0^{R_2},$$

exists at the point (p, q), *when* $R_1, R_2 \to \infty$.

Some important special cases of (1.1), the inversion formula and certain basic properties for our H-H integral transform have been discussed in detail in the aforementioned paper by the author.

In this paper, we prove three interesting theorems giving the relationships of the H-H integral transform defined by (1.1) with Meijer-Laplace transform, Mellin and Laplace transform of two variables. One of these theorems has been used to evaluate a general double integral.

It may be noted that the parameters of all the H-functions of two variables occurring in this paper are same as those of the H-function of two variables involved in (1.1). Therefore to simplify the space problem we use $H_1[x, y]$ to denote the H-function of two variables throughout the paper.

2. Relationship with meijer-Laplace transform

In 1959, Bhise [12] studied an elegant unification of several known generalization of the classical Laplace transform in the form (see also [3]):

$$G\{F(v):u\}=u\int_0^\infty G_{M,\ M+1}^{M+1,\ 0}\left(uv\left|{(\lambda_M+\xi_M) \atop (\lambda_M),\ \gamma}\right.\right)F(v)\ dv \cdots\cdots (2.1)$$

where $G_{p,q}^{m,n}(x)$ is the well-known Meijer G-function and if the function $F(v)$ is piecewise continuous in every finite interval $0<v\leqq R, R>0$, and

$$F(v)=\begin{cases}O(x^{\delta}e^{-\rho v}),\ \mathrm{Re}(\rho)>0,\ v\to\infty \\ O(v^{\sigma}),\ v\downarrow 0.\end{cases} \cdots\cdots (2.2)$$

then for existence of Meijer-Laplace transform (2.1),

$$\mathrm{Re}(u)>0,\ \mathrm{Re}(\sigma)+\min_{1\leqq j\leqq M}\{\mathrm{Re}(\lambda_j),\ \mathrm{Re}(\gamma)\}>0. \cdots\cdots (2.3)$$

The following theorem provides interesting relationship between the Meijer-Laplace transform (2.1) and the H-H integral transform (1.1).

THEOREM 1. *Suppose that the two-dimensional H-H integral transform* $|H\text{-}H\{f:p,q\}|$ *defined by* (1.1) *exists. Also, let the function* $F(v)$ *be piecewise continuous in every finite interval* $(0,\ R]$, $R>0$, *and have the asymptotic expansions given by* (2.2).
Then

$$G\{F(v)\ H\text{-}H\{f:p,\ q\}:u\}$$

$$=\int_0^\infty\int_0^\infty f\left(\frac{x}{p},\ \frac{y}{q}\right)dx\ dy$$

$$\cdot G\left\{v^2F(v)H_{s,t}^{r,0}\left[axv\left|{(k_j,\mu_j)_{1,s} \atop (u_j,\ \nu_j)_{1,t}}\right.\right]H_{S,T}^{R,0}\left[byv\left|{(g_j,G_j)_{1,S} \atop (h_j,H_j)_{1,T}}\right.\right]H_1[mxv,\ nyv]:u\right\} \cdots\cdots (2.4)$$

provided that the double integrals in (2.4) *converge absolutely.*

$\mathrm{Re}(p)>0$, $\mathrm{Re}(q)>0$, $\mathrm{Re}(u)>0$, *and*

$$\mathrm{Re}(\sigma+3)+\min_{1\leqq j\leqq M}\{\mathrm{Re}(\lambda_j),\ \mathrm{Re}(\gamma)\}+\alpha+\alpha'+\beta+\beta'>0 \cdots\cdots (2.5)$$

where α, α', β *and* β' *are given by* (1.6) *and* (1.7) *respectively.*

PROOF. In order to establish the relationship (2.4), we start with the definitions (1.1) and (2.1) to get

$$G\{F(v)\ H\text{-}H\ \{f:pv,\ qv\}:u\}$$

$$=u\int_0^\infty G_{M,\ M+1}^{M+1,\ 0}\left(uv\left|{\lambda_M+\xi_M) \atop (\lambda_M),\ \gamma}\right.\right)F(v)\left[pqv^2\int_0^\infty\int_0^\infty H_{s,t}^{r,0}\ [apxv]\right.$$

$$\cdot H_{S,T}^{R,0}[bqyv]\ H_1[mxpv,\ nqyv]\ f(x,y)\ dx\ dy\Big]\ dv$$

$$=\int_0^\infty\int_0^\infty f\Big(\frac{x}{p},\ \frac{y}{q}\Big)\Big[u\int_0^\infty G_{M,\ M+1}^{M+1,\ 0}\Big(uv\Big|{}^{(\lambda_M+\xi_M)}_{(\lambda_M),\ \gamma}\Big)\ v^{2}\ F(v)$$

$$\cdot H_{s,t}^{r,0}\ [axv]\ H_{S,T}^{R,0}\ [byv]\ H_1\ [mxv,\ nyv]\ dv\Big]\ dx\ dy.$$

The final result (2.4) would follow immediately if we interpret the v-integral by means of (2.1). The inversion of order of integration in the above prove is justified under the conditions stated above.

It is remarked here that for $s=S=M=0$, $r=t=R=T=1$, $u_1=h_1=0$, $\nu_1=H_1=1$, our result (2.4) corresponds essentially to the relationship given by Goyal and Vasistha ([6], Eq. (3.1)).

A special case of the Theorem 1, when $F(v)=v^{\xi}$, $v\geqq 0$, is worthy of note. Indeed, by appealing a known result which is a particular case of a integral given by Goyal ([5], p. 39, Eq. (4.4)) we find that

$$G\Big\{v^{\xi}\ H_{s,t}^{r,0}\Big[axv\Big|{}^{(k_j,\mu_j)_{1,s}}_{(u_j,\nu_j)_{1,t}}\Big]\ H_{S,T}^{R,0}\Big[byv\Big|{}^{(g_j,G_j)_{1,S}}_{(h_j,H_j)_{1,T}}\Big]H_1[mxv,\ nyv]\ :u\Big\}$$

$$=u^{-\xi}\sum_{N=1}^{r}\sum_{P=1}^{R}\sum_{U=0}^{\infty}\sum_{V=0}^{\infty}\frac{(-1)^{U+V}\ (ax/u)^{\rho_U}\ (by/u)^{\sigma_V}\ \theta_1(\rho_U)\ \theta_2(\sigma_V)}{U!\ \ V!\qquad \nu_N\ \ H_P}$$

$$\cdot H_{P_1+M+1,\ q_1+M\ :\ p_2,\ q_2;\ p_3,\ q_3}^{0,\qquad M+1\ :\ m_2,\ n_2;\ m_3,\ n_3}\left[{}^{\mathrm{mx/u}}_{\mathrm{ny/u}}\Big|{}^{(-\lambda_j-\xi-\rho_U-\sigma_V;\ 1,1)_{1,M},\ (-\gamma-\xi-\rho_U-\sigma_V;\ 1,\ 1),\ (a_j;\alpha_j,\ A_j)_{1,p_1}:\ (c_j,\ \varepsilon_j)_{1,p_2};\ (e_j,\ E_j)_{1,p_3}}_{(-\lambda_j-\xi_j-\xi-\rho_U-\sigma_V;\ 1,1)_{1,M},\ (b_j;\beta_j,\ B_j)_{1,q_1}\ :\ (d_j,\ \delta_j)_{1,q_2};\ (f_j,\ F_j)_{1,q_3}}\right]$$

$$=\Psi(\xi,\ a,\ b,\ m,\ n,\ e_U,\ \sigma_V;\ x,\ y)\ \cdots\cdots\cdots\cdots(2.6)$$

where

$$\rho_U=\frac{U_N+U}{\nu_N},\ \forall N\epsilon\{1,\ \cdots,\ r\}\ \text{and}\ U=0,\ 1,\ 2\cdots;\ \sigma_V=\frac{h_P+V}{H_P},\ \forall F\epsilon\ \{1,\ \cdots,\ R\}\ \text{and}$$

$$V=0,\ 1,\ 2,\ \cdots,$$

$$\theta_1(\rho_U)=\prod_{\substack{j=1\\ j\neq N}}^{r}\Gamma(u_j-\nu_j\rho_U)\Big[\prod_{j=r+1}^{t}\Gamma(1-u_j+\nu_j\rho_U)\prod_{j=1}^{s}\Gamma(k_j-\mu_j\rho_U)\Big]^{-1},\ \cdots\cdots\cdots(2.7)$$

$$\theta_2(\sigma_V)=\prod_{\substack{j=1\\ j\neq P}}^{R}\Gamma(h_j-H_j\sigma_V)\Big[\prod_{j=R+1}^{T}\Gamma(1-h_j+H_j\sigma_V)\ \prod_{j=1}^{S}\Gamma(g_j-G_j\sigma_V)\Big]^{-1}\cdots\cdots(2.8)$$

Integral (2.6) is valid under the following conditions.

$\mathrm{Re}(u)>0$, $S_1>0$, $S_2>0$, $\Lambda_1>0$, $\Lambda_2>0$,

$$|\arg ax| < \frac{1}{2} S_1 \pi, \quad |\arg by| < \frac{1}{2} S_2 \pi, \quad |\arg mx| < \frac{1}{2} \Lambda_1 \pi \text{ and } |\arg ny| < \frac{1}{2} \Lambda_2 \pi, \qquad \cdots\cdots (2.9)$$

$$\mathrm{Re}(\xi+1) + \min_{1 \leq j \leq M} \{\mathrm{Re}(\lambda_j),\ \mathrm{Re}(\gamma)\} + \alpha + \alpha' + \beta + \beta' > 0,$$

where S_1, S_2, Λ_1, Λ_2 are defined by (1.2) and (1.4) respectively, while α, α', β, and β' are given by (1.6) and (1.7) respectively.
Thus we have

COROLLARY. *Assuming that the inequalities in* (2.9) *hold when* ξ *is replaced by* $\xi+2$, *let* $\mathrm{Re}(p)>0$, $\mathrm{Re}(q)>0$.
Then

$$G\{v^{\xi}\ H\text{-}H\ \{f : pv,\ qv\} : u\}$$
$$= \int_0^\infty \int_0^\infty f\left(\frac{x}{p},\ \frac{y}{q}\right) \Psi(\xi+2,\ a,\ b,\ m,\ n,\ \rho_U,\ \sigma_V;\ x,\ y)\, dx\, dy \quad \cdots\cdots (2.10)$$

where $\Psi(\ldots)$ *is defined by* (2.6) *and double integrals in* (2.10) *converge absolutely.*

3. Relationship with double mellin transform

The following theorem gives the connection of double Mellin transform with H-H integral transform defined by (1.1).

THEOREM 2. *Let the two-dimensional Mellin transform of* $|f(x, y)|$ *and* $|\phi(p,q)|$ *exist, and define* α, α', β *and* β' *by the equations* (1.6) *and* (1.7) *respectively.*

Then

$$\bar{\phi}(u, v) = \Phi(1+u,\ 1+v) \bar{f}(-u,\ -v) \cdots\cdots (3.1)$$

provided that H-H integral transform $|\phi(p, q)|$ *defined by* (1.1) *exists, and*

$$\mathrm{Re}(u) + \alpha + \alpha' + 1 > 0, \quad \mathrm{Re}(v) + \beta + \beta' + 1 > 0 \cdots\cdots (3.2)$$

Also, for convenience, $\bar{f}(u, v)$ *denotes the double Mellin transform of* $f(x, y)$, *which is defined by the following equation*:

$$\bar{f}(u, v) = \int_0^\infty \int_0^\infty x^{u-1}\, y^{v-1}\, f(x, y)\, dx\, dy \cdots\cdots (3.3)$$

and

$$\Phi(u, v) = a^{-u} b^{-v} H^{0,\ 0\ :\ m_2,\ n_2+r;\ m_3,\ n_3+R}_{p_1,\ q_1\ :\ p_2+t,\ q_2+s;\ p_3+T,\ q_3+S}\left[\begin{matrix} m/a \\ n/b \end{matrix} \middle| \begin{matrix} (a_j; \alpha_j, A_j)_{1,p_1} : (c_j, \varepsilon_j)_{1,n_2},\ (1-u_j-\nu_j u, \\ (b_j; \beta_j, B_j)_{1,q_1} : (d_j, \delta_j)_{1,q_2},\ (1-k_j-\mu_j u, \end{matrix}\right.$$

$$\left.\begin{matrix} \nu_j)_{1,t}, (c_j, \varepsilon_j)_{n_2+1,p_2}; (e_j, E_j)_{1,n_3}, (1-h_j-H_j v, H_j)_{1,T}, (e_j, E_j)_{n_3+1,p_3} \\ \mu_j)_{1,s}; \qquad (f_j, F_j)_{1,q_3}, (1-g_j-G_j v, G_j)_{1,S} \end{matrix}\right] \cdots (3.4)$$

PROOF. We have by virtue of (1.1) and (3.3)

$$\bar{\phi}(u, v) = \int_0^\infty\int_0^\infty p^{u-1} q^{v-1} \Big\{ pq \int_0^\infty\int_0^\infty H_{s,t}^{r,0}[apx]\, H_{S,T}^{R,0}[bqy]$$

$$\cdot H_1[mpx, nqy]\, f(x, y)\, dx\, dy \Big\}\, dp\, dq$$

$$= \int_0^\infty\int_0^\infty f(x, y) \Big\{ \int_0^\infty\int_0^\infty p^u q^v H_{s,t}^{r,0}[apx]\, H_{S,T}^{R,0}[bqy]$$

$$\cdot H_1[mpx, nqy]\, dp\, dq \Big\}\, dx\, dy \cdots\cdots (3.5)$$

where the change of order of integration is permissible due to conditions imposed with Theorem 2. Now evaluating the inner p, q-integral in (3.5) with the help of the following result, which is a particular case of an integral given by Handa ([11], p. 98, Eq. (2.2.1)):

$$\int_0^\infty\int_0^\infty x^{\rho-1} y^{\sigma-1} H_{s,t}^{r,0}\left[apx \,\middle|\, \begin{matrix}(k_j, \mu_j)_{1,s} \\ (u_j, \nu_j)_{1,t}\end{matrix}\right] H_{S,T}^{R,0}\left[bqy \,\middle|\, \begin{matrix}(g_j, G_j)_{1,S} \\ (h_j, H_j)_{1,T}\end{matrix}\right]$$

$$\cdot H_1[mpx, nqy]\, dx\, dy$$

$$= p^{-\rho} q^{-\sigma}\, \Phi(\rho, \sigma) \cdots\cdots (3.6)$$

where $\Phi(\rho, \sigma)$ is defined by (3.4) and the integral (3.6) is valid under the following conditions.

(i) The set of conditions (i) to (iv) given in section 1 are satisfied.

(ii) $\mathrm{Re}(\rho)+\alpha+\alpha'>0$, $\mathrm{Re}(\sigma)+\beta+\beta'>0$, where α, α', β and β' are defined by (1.6) and (1.7) respectively.

we easily get the (3.1) with the help of (3.3).

Evidently the functional relationship (3.1), when viewed in conjuction with Reed's Theorem ([14], p. 566) would lead us at once to inversion formula for our transform (1.1) ([1], Theorem 1).

If we reduce two Fox's H-functions involved in (1.1) to exponential functions with the help of the relation ([12], p. 10, Eq. (1.2.2)) in the above theorm, we arrive at the known result due to Gupta, Garg and Kalla ([8], p. 58, Eq. (1.2)).

4. Relationship with double Laplace transform

The well-known double Laplace transform $f(x, y)$ of any function $g(u, v)$ is given by the following integral equation:

$$f(x, y)=\int_0^\infty\int_0^\infty e^{-xu-yv}\ g(u, v)\ du\ dv \cdots\cdots(4.1)$$

$$\{\mathrm{Re}(x)>0,\ \mathrm{Re}(y)>0\}$$

Following theorem will give us the relationship between the H-H integral transform and double Laplace transform:

THEOREM 3. *Let $f(x, y)$ be given by the equation (4.1) and $\Phi(u, v)$ is defined by (3.4). Also, suppose that the two-dimensional H-H integral transform of $|f(x,y)|$ defined by (1.1) exists.*

Then

$$H\text{-}H\{f : p, q\} = \sum_{K,L=0}^{\infty} \frac{(-1)^{K+L}}{p^K q^L\ K!\ L!}\ \Phi(1+K,\ 1+L) \int_0^\infty\int_0^\infty u^K\ v^L\ g(u,v)\ du\ dv \cdots\cdots(4.2)$$

provided that

(i) *the double series on the right-hand side of* (4.2) *is absolutely convergent.*

(ii) $\mathrm{Re}(x)>0$, $\mathrm{Re}(y)>0$, $\alpha+\alpha'+1>0$, *and* $\beta+\beta'+1>0$, *where* α,α', *and* β,β' *are given by* (1.6) *and* (1.7) *respectively.*

PROOF. Substituting the value of $f(x,y)$ from (4.1) in (4.2) and changing the order of integrations, which is justified due to absolute convergence of the integrals involved, we find that

$$H\text{-}H\{f : p, q\} = \int_0^\infty\int_0^\infty g(u, v) \Big[pq \int_0^\infty\int_0^\infty e^{-ux-vy}\ H_{s,t}^{r,0}\ [apx]$$

$$\cdot H_{S,T}^{R,0}\ [bqy]\ H_1[mpx,\ nqy]\ dx\ dy\Big]\ du\ dv.$$

On using the series expansion for e^{-ux-vy} and (3.6), we get the desired result (4.2).

Evidently, the relationship given by Gupta, Garg and Kalla ([8], p. 64, Theorem 2) is contained in Theorem 3 as a particular case.

APPLICATION OF THEOREM 3. Let

$$g(u, v)=H_{p_1', q_1'}^{m_1', 0}\left[z_1 u \,\Big|\, \begin{matrix}(c_j', \varepsilon_j')_{1,p_1'}\\(d_j', \delta_j')_{1,q_1'}\end{matrix}\right] H_{p_2', q_2'}^{m_2', 0}\left[z_2 v \,\Big|\, \begin{matrix}(e_j', E'_j)_{1,p_2'}\\(f_j', F_j')_{1,q_2'}\end{matrix}\right], \cdots\cdots(4.3)$$

then with the help of (4.1) and a known result ([10], p. 190, Eq. (2.4)), we find that

$$f(x, y)=\frac{1}{xy} H^{m_1', 1}_{p_1'+1, q_1'}\left[z_1 x^{-1}\middle|\begin{matrix}(0,1), (c_j', \epsilon_j')_{1,p_1'}\\ (d_j', \delta_j')_{1,q_1'}\end{matrix}\right]$$

$$\cdot H^{m_2', 1}_{p_2'+1, q_2'}\left[z_2 y^{-1}\middle|\begin{matrix}(0,1), (e_j', E_j')_{1,p_2'}\\ (f_j', F_j')_{1,q_2'}\end{matrix}\right]. \quad \cdots\cdots(4.4)$$

(4.4) is valid for

$\mathrm{Re}(x, y)>0$, $S_i'>0$ $(i=1,2)$, where

$$S_1'=\sum_{j=1}^{m_1'}\delta_j'-\sum_{j=m_1'+1}^{q_1'}\delta_j'-\sum_{j=1}^{p_1'}\epsilon_j'>0,$$

$$S_2'=\sum_{j=1}^{m_2'}F_j'-\sum_{j=m_2'+1}^{q_2'}F_j'-\sum_{j=1}^{p_2'}E_j'>0,$$

$$|\arg(z_i)|<\frac{1}{2}S_i'\Pi(i=1,2)$$

$$\min_{1\leq j\leq m_1'}\{\mathrm{Re}(d_j'/\delta_j')\}>-1, \text{ and } \min_{1\leq j\leq m_2'}\{\mathrm{Re}(f_j'/F_j')\}>-1.$$

Also, with the help of well-known Mellin inversion formula;

$$\int_0^\infty\int_0^\infty u^K\ v^L\ H^{m_1', 0}_{p_1', q_1'}\left[z_1 u\middle|\begin{matrix}(c_j', \epsilon_j')_{1,p_1'}\\ (d_j', \delta_j')_{1,q_1'}\end{matrix}\right] H^{m_2', 0}_{p_2', q_2'}\left[z_2 v\middle|\begin{matrix}(e_j', E_j')_{1,p_2'}\\ (f_j', F_j')_{1,q_2'}\end{matrix}\right] du\ dv$$

$$=(z_1)^{-(1+K)}(z_2)^{-(1+L)}\prod_{j=1}^{m_1'}\Gamma(d_j'+\delta_j'+\delta_j'K)\prod_{j=1}^{m_2'}\Gamma(f_j'+F_j'+F_j'L)$$

$$\cdot\Big[\prod_{j=m_1'+1}^{q_1'}\Gamma(1-d_j'-\delta_j'-\delta_j'K)\prod_{j=1}^{p_1'}\Gamma(c_j'+\epsilon_j'+\epsilon_j'K)$$

$$\cdot\prod_{j=m_2'+1}^{q_2'}\Gamma(1-f_j'-F_j'-F_j'L)\prod_{j=1}^{p_2'}\Gamma(e_j'+E_j'+E_j'L)\Big]^{-1} \quad\cdots\cdots(4.5)$$

provided that the conditions except $\mathrm{Re}(x, y)>0$ mentioned with (4.4) are satisfied.

On putting the values given by (4.4) and (4.5) respectively in the relationship (4.2) and using the known properties (see e.g. [12], p.4, Eqs. (1.2.4) and (1.2.2)) for the Fox's H-functions

$$H^{m_1', 1}_{p_1'+1, q_1'}[z_1 x^{-1}] \text{ and } H^{m_2', 1}_{p_2'+1, q_2'}[z_2 y^{-1}]$$

therein, and adjusting the parameters suitably, we get the following general double integral, which is believed to be new.

$$\int_0^\infty\int_0^\infty H^{r,0}_{s,t}\left[ax\middle|\begin{matrix}(k_j, \mu_j)_{1,s}\\ (u_j, \nu_j)_{1,t}\end{matrix}\right] H^{R,0}_{S,T}\left[by\middle|\begin{matrix}(g_j, G_j)_{1,S}\\ (h_j, H_j)_{1,T}\end{matrix}\right]$$

$$\cdot H^{1, n_1'}_{p_1', q_1'+1}\left[z_1 x\middle|\begin{matrix}(c_j', \epsilon_j')_{1,p_1'}\\ (0,1), (d_j', \delta_j')_{1,q_1'}\end{matrix}\right] H^{1, n_2'}_{p_2', q_2'+1}\left[z_2 y\middle|\begin{matrix}(e_j', E_j')_{1,p_2'}\\ (0,1), (f_j', F_j')_{1,q_2'}\end{matrix}\right]$$

$$\cdot H_1[mx, ny]\ dx\ dy$$

$$= \sum_{K=0}^{\infty} \sum_{L=0}^{\infty} \frac{(-z_1)^K (-z_2)^L}{K!\ L!} \theta_1'(K)\theta_2'(L)\, \Phi(1+K, 1+L) \cdots\cdots\cdots (4.6)$$

where

$$\theta_1'(K) + \prod_{j=1}^{n_1'} \Gamma(1-c_j'+\epsilon_j'K)\Big[\prod_{j=1}^{q_1'} \Gamma(1-d_j'+\delta_j'K) \prod_{j=n_1'+1}^{p_1'} \Gamma(c_j'-\epsilon_j'K)\Big]^{-1} \quad (4.7)$$

and $\theta_2'(L)$ can be written analogously to $\theta_1'(K)$ in terms of the parameter sets $(e_j', E_j')_{1,p_2'}$ and $(f_j', F_j')_{1,q_2'}$. Also $\Phi(K, L)$ is defined by (3.4).

The conditions for validity of (4.6) are

(i) The inequalities (1.2) to (1.5) (with $p=q=1$) are satisfied.

(ii) $$S_3 = \sum_{j=1}^{n_1'} \epsilon_j' - \sum_{j=n_1'+1}^{p_1'} \epsilon_j' - \sum_{j=1}^{q_1'} \delta_j' + 1 > 0,$$

$$S_4 = \sum_{j=1}^{n_2'} E_j' - \sum_{j=n_2'+1}^{p_2'} E_j' - \sum_{j=1}^{q_2'} F_j' + 1 > 0,$$

$|\arg z_1| < \frac{1}{2} S_3 \Pi$ and $|\arg z_2| < \frac{1}{2} S_4 \Pi$

(iii) $\alpha+\alpha'+1>0$, and $\beta+\beta'+1>0$ (α, α' and β, β' are given by (1.6) and (1.7) respectively).

The double integral (4.6) is quiete general in nature and a number of (known and unknown) single and double integrals can be derived as its special cases. For the sake of illustration, we choose to mention only one special case of (4.6) and leave others as an fruitful exercise for the intersted reader. Indeed, if in (4.6) we take $n_1'=p_1'=q_1'=n_2'=p_2'=q_2'=0$, we easily get the following interesting double integral with the help of known relation ([12], p. 10, Eq. (1.7.2)).

$$\int_0^\infty \int_0^\infty e^{-z_1x-z_2y}\, H_{s,t}^{r,0}\left[a\,x \,\middle|\, \begin{matrix}(k_j, \mu_j)_{1,s}\\(u_j, \nu_j)_{1,t}\end{matrix}\right] H_{S,T}^{R,0}\left[b\,y \,\middle|\, \begin{matrix}(g_j, G_j)_{1,S}\\(h_j, H_j)_{1,T}\end{matrix}\right]$$

$$\cdot H_1[mx, ny]\ dx\ dy$$

$$= \sum_{K,L=0}^{\infty} \frac{(-z_1)^K\ (-z_2)^L}{K!\quad L!}\ \Phi(1+K,\ 1+L) \cdots\cdots\cdots\cdots\cdots\cdots (4.8)$$

where the conditions easily obtainable from those given with (4.6) are satisfied.

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