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ABSOLUTE RIESZ SUMMABILITY OF SERIES ASSOCIATED WITH LEGENDRE SERIES

By Prem Chandra and M. M. Sharma

1. Definitions and notations

Let $\lambda=\lambda(w)$ be a continuous, differentiable and monotonic increasing function of w and let it tend to infinity with w. Suppose that $\sum_{n=1}^{\infty} u_n$ be a given infinite series, then

$$\sum_{n=1}^{\infty} u_n \in |R,\ \lambda(w),\ r|\quad (r>0)\ \text{if}$$

$$\int_{l}^{\infty} \frac{\lambda^{(1)}(w)}{\{\lambda(w)\}^{\lambda+1}} \left| \sum_{n\leq w} \{\lambda(w)-\lambda(n)\}^{r-1} \lambda(n) u_n \right| dw < \infty$$

where l is a suitable finite number (Obrechkoff [3], [4])

and $$\lambda^{(1)}(w)=\frac{d}{dw}(\lambda(w)).$$

Let f be a Lebesgue-measurable function over the linear interval $[-1,\ 1]$. Then its Legendre series at $x\in[-1,\ 1]$ is given by

$$\sum_{n=1}^{\infty} a_n\ P_n(x)$$

where

$$a_n=\left(n+\frac{1}{2}\right)\int_{-1}^{1} f(y)\ P_n(y)\ dy \tag{1.1}$$

and $P_n(x)$ denotes the n-th Legendre polynomial.

Throughout the paper we take fixed x such that $-1<x<1$. We also write

$$f_1(t)=\frac{1}{t}\int_0^t f(u)\ du.$$

2. Introduction

Recently Pal and Sharma [5] and Chandra and Sharma [1] have studied absolute Riesz summability of Legendre series of a function of bounded variation. In this paper, we study absolute Riesz summability factors for Legendre series at an internal point of $[-1, 1]$ under the local conditions. Precisely, we prove the following

THEOREM. *Let $h(t)$ be non-negative and non-decreasing with $t \geq 0$ and let*

$$\int_t^{\pi} \phi^{-1} h(\phi)\, d\phi = O\left\{\mu\left(\frac{1}{t}\right)\right\} \quad (t \to 0+), \tag{2.1}$$

where μ is positive and non-decreasing. Then

$$\alpha(t) = \int_0^t |f(\pm \cos u)|\; du = O(h(t)\sqrt{t})(t \to 0+) \tag{2.2}$$

and

$$\beta(t) = \int_0^t |f(\cos(\theta \pm u))|\, du = O(t) \quad (t \to 0+) \tag{2.3}$$

imply $\sum_{n=1}^{\infty} a_n P_n(x)\; y(n) \in |R, \lambda(w), 1|$, *whenever*

$$\{\lambda(n)\; y(n)\} \downarrow \textit{ for } n \geq n_0 \tag{2.4}$$

$$\sum_{n \leq w} \lambda(n) y(n)/n = O\{\lambda^2(w) y(w)/w\; \lambda^{(1)}(w)\} \tag{2.5}$$

$$\text{(2.6)} \begin{cases} \text{(i)} & \displaystyle\int_l^{\infty} \frac{\lambda^{(1)}(w)}{\lambda(w)}\, y(w) \log w\; dw < \infty; \\ \text{(ii)} & \displaystyle\int_l^{\infty} \frac{y(w)\; \mu(w)}{w}\, dw < \infty. \end{cases}$$

We remark that

$$f(t) \in BV[-1, 1] \Longrightarrow f_1(t) \in BV[-1, 1]$$

and the latter implies that

$$\int_0^t |f(u)|\; du = O(t) \quad (t \to 0),$$

but the converse, in general, is not true (Compare with Chandra and Gupta [2]). Therefore, if

$$h(t) = \sqrt{t}$$

the hypotheses (2.2) and (2.3) are weaker than

$$f_1(t) \in BV[-1, 1].$$

3. Lemmas

We shall use the following lemmas in the proof:

LEMMA 1. ([6], p. 197). *Let* $0<\theta<\pi$ *and* $n=1, 2, 3, \cdots$. *Then*

$$|P_n(\cos\theta)|\leq 4\sqrt{\frac{2}{n\pi\sin\theta}}.$$

LEMMA 2. *Let* $n=1, 2, 3, \cdots$ *and* c *be a fixed positive number. Then, for* $c/n \leq\theta\leq\pi-c/n$,

$$P_n(\cos\theta)=\sqrt{\frac{2}{n\pi\sin\theta}}\cos\left\{\left(n+\frac{1}{2}\right)\theta-\pi/4\right\}+O\left\{\frac{n^{-3/2}}{(\sin\theta)^{3/2}}\right\}.$$

This follows from (8.21.18) of Szego [7].

LEMMA 3. *Let* c *be a fixed positive number and* $c/n\leq\theta\leq\pi-\frac{c}{n}$ *and* $c/n\leq\phi\leq \pi-c/n$. *Then for* $n\geq1$

$$\begin{aligned}&P_n(\cos\theta)\ P_n(\cos\phi)\ \sin\phi\\ &=\frac{1}{n\pi}\sqrt{\frac{\sin\phi}{\sin\theta}}\left\{\sin\left(n+\frac{1}{2}\right)(\theta+\phi)+\cos\left(n+\frac{1}{2}\right)(\theta-\phi)\right\}\\ &\qquad+O\left\{\frac{n^{-2}}{\sqrt{(\sin\phi)}}\right\}.\end{aligned}$$

This follows from Lemma 2 after some simplification.

4. Proof of the theorem

The series $\sum_{n=1}^{\infty}a_n P_n(x)\ y(n)\in|R,\ \lambda(w),\ 1|$ *if*

$$\int_l^{\infty}\frac{\lambda^{(1)}(w)}{(\lambda(w))^2}\left|\sum_{n\leq w}\lambda(n)\ a_n\ P_n(x)\ y(n)\right|\,dw<\infty,$$

where l *is some positive constant, and, by* (1.1),

$$\begin{aligned}a_nP_n(x)&=\left(n+\frac{1}{2}\right)\int_0^{\pi}f(\cos\phi)\ P_n(\cos\theta)\ P_n(\cos\phi)\ \sin\phi\, d\phi\\ &=\left(n+\frac{1}{2}\right)\left(\int_0^{c/w}+\int_{c/w}^{\theta-c/w}+\int_{\theta-c/w}^{\theta+c/w}+\int_{\theta+c/w}^{\pi-c/w}+\int_{\pi-c/w}^{\pi}\right)\\ &=\sum_{r=1}^{5}I_r(n),\ \text{say},\end{aligned}$$

where constant $c\geq 1$.

Thus, for the proof of the theorem, it is enough to show that

$$(4.1)\qquad \int_{l}^{\infty}\frac{\lambda^{(1)}(w)}{(\lambda(w))^{2}}\left|\sum_{n\leq w}\lambda(n)\ y(n)\ I_{r}(n)\right|\ dw<\infty$$

for $r=1, 2, 3, 4,$ and 5.

By Lemma 1,

$$\begin{aligned} I_1(n) &= O\ (1)\int_0^{c/w}\phi^{1/2}|f(\cos\phi)|\ d\phi \\ &= O\ (w^{-1}h(c/w)) \qquad (by(2.2)) \\ &= o\left(\frac{1}{n}\right), \end{aligned}$$

for $n\leq w$. By using the transformation

$$u=\pi-\phi$$

and proceeding as above, it may be shown that

$$I_5(n)=o\left(\frac{1}{n}\right),$$

for $n\leq w$. Now, we consider $I_3(n)$.

By Lemma 3,

$$\begin{aligned} I_3(n) = {} & \left(\frac{2n+1}{2n\pi}\right)\int_{\theta-c/w}^{\theta+c/w} f\ (\cos\phi)\ \left(\frac{\sin\phi}{\sin\theta}\right)^{1/2}\Big[\sin\Big(n+\frac{1}{2}\Big)(\theta+\phi) \\ & \qquad +\cos\Big(n+\frac{1}{2}\Big)(\theta-\phi)\Big]\ d\phi \\ & +O\left(\frac{1}{n}\right)\int_{\theta-c/w}^{\theta+c/w}\phi^{-1/2}\ |f(\cos\phi)|\ d\phi \\ = {} & O(1)\int_{\theta-c/w}^{\theta+c/w}|f(\cos\phi)|\ d\phi+O\left(\frac{1}{n}\right) \\ = {} & O(1)\int_{-c/w}^{c/w}|f(\cos(\theta-u))|\ du+O\left(\frac{1}{n}\right) \\ = {} & o\left(\frac{1}{n}\right), \end{aligned}$$

by (2,3), for $n\leq w$. Thus

$$I_r(n)=o\left(\frac{1}{n}\right)\quad (r=1,\ 3,\ 5).$$

Now, using (2.5) and (2.6) (ii), (4.1) for r=1, 3, 5 holds.

Since the proof of (4.1) for $r=4$ is similar to that for $r=2$, therefore we prove (4.1) for $r=2$ only.

By Lemma 3,

$$I_2(n)=\frac{1}{\pi}\int_{c/w}^{\theta-c/w} f(\cos\phi)\left(\frac{\sin\phi}{\sin\theta}\right)^{1/2}\Big[\sin\Big(n+\frac{1}{2}\Big)(\theta+\phi)$$

$$+\cos\Big(n+\frac{1}{2}\Big)(\theta-\phi)\Big]\,d\phi+O\Big(\frac{1}{n}\Big)\int_{c/w}^{\theta-c/w}|f(\cos\phi)|\,\phi^{-1/2}\,d\phi$$

$$=\frac{1}{\pi}I_{2,1}(n)+I_{2,2}(n),\ \text{say.}$$

Now, integrating by parts, we have

$$\int_{c/w}^{\theta-c/w}|f(\cos\phi)|\,\phi^{-1/2}\,d\phi=[\phi^{-1/2}\alpha(\phi)]_{c/w}^{\theta-c/w}+\frac{1}{2}\int_{c/w}^{\theta-c/w}\phi^{-\frac{3}{2}}\alpha(\phi)\,d\phi$$

$$=O(1)+O(1)\int_{c/w}^{\theta-c/w}\frac{h(\phi)}{\phi}\,d\phi$$

(by (2.2))

$$=O(1)+O(\mu(w/c))$$

(by (2.1))

$$=O(\mu(w)).$$

Therefore

$$I_{2,2}(n)=O\Big\{\frac{\mu(w)}{n}\Big\}$$

and hence (4.1) with $I_{2,2}(n)$ in place of $I_2(n)$ holds whenever we use (2.5) and (2.6) (ii). Finally, to complete the proof of the theorem it remains to show that (4.1) holds with $I_{2,1}(n)$ for $I_2(n)$.

Now

$$\sum_{n\le w}\lambda(n)\,y(n)\,I_{2,1}(n)$$

$$=\int_{c/w}^{\theta-c/w} f(\cos\phi)\left(\frac{\sin\phi}{\sin\theta}\right)^{1/2}d\phi\Big(\sum_{n\le w}\lambda(n)\,y(n).$$

$$\cdot\Big\{\sin\Big(n+\frac{1}{2}\Big)(\theta+\phi)+\cos\Big(n+\frac{1}{2}\Big)(\theta-\phi)\Big\}\Big)$$

$$=O\Big\{\lambda(w)\,y(w)\int_{c/w}^{\theta-c/w}\frac{|f(\cos\phi)|}{\theta-\phi}\,d\phi\Big\}$$

and integrating by parts

$$\int_{c/w}^{\theta-c/w} \frac{|f(\cos\phi)|}{\theta-\phi}\, d\phi = \int_{c/w}^{\theta-c/w} \frac{|f(\cos(\theta-u))|}{u}\, du$$
$$= O(\log w),$$

by (2.3). Therefore

$$\sum_{n\leqq w} \lambda(n)\ y(n)\ I_{2,1}(n) = O\{\lambda(w)\ y(w)\ \log w\}.$$

Hence, in view of (2.6) (i), (4.1) with $I_{2,1}(n)$ for $I_2(n)$ holds.

This completes the proof of the theorem.

5. Corollaries

We deduce from the theorem the following corollaries:

COROLLARY 1. *Let* $\Delta \geq 1$ *and let*

(5.1) (i) $\alpha(t)=0(\sqrt{t})(t\to 0+)$; (ii) $\beta(t)=O(t)$ $(t\to 0)$.

Then

$$\sum_{n=1}^{\infty} \frac{a_n P_n(x)}{\{\log(n+1)\}^{1+\varepsilon+\Delta}} \in |R,\ \exp\{(\log w)^{\Delta}\},\ 1|\ \ (\varepsilon>0).$$

PROOF. Let $h(t)=1$ and $\lambda(w)=\exp\{(\log w)^{\Delta}\}$ $(\Delta\geq 1)$ in the theorem. Then

$$\mu_n = \log n$$

and the conditions (2.4), (2.5) and (2.6) are satisfied for

$$y_n = \{\log(n+1)\}^{-1-\varepsilon-\Delta}\ (\varepsilon>0).$$

Thus the proof of Corollary 1 follows.

COROLLARY 2. *Let* $0<\alpha<1$ *and* $\varepsilon>0$ *and let* (5.1) *hold. Then*

$$\sum_{n=1}^{\infty} \frac{a_n P_n(x)}{n^{\alpha}(\log(n+1))^{2+\varepsilon}} \in |R,\ \exp(w^{\alpha}),\ 1|.$$

PROOF. By taking $h(t)=1$, $\lambda(w)=\exp(w^{\alpha})$ $(0<\alpha<1)$ and $y(n)=(\log(n+1))^{-2-\varepsilon}n^{-\alpha}$ in the theorem, the proof of the corollary follows.

COROLLARY 3. *Let* $\varepsilon>0$ *and let*

(5.2) (i) $\alpha(t)=O\left(\dfrac{\sqrt{t}}{(\log\frac{1}{t})^{1+\varepsilon}}\right)$ $(t\to 0)$; (ii) $\beta(t)=O(t)$ $(t\to 0)$.

Then

$$\sum_{n=1}^{\infty} \frac{a_n P_n(x)}{\{\log(n+1)\}^{1+\varepsilon}} \in |R, \log w, 1|.$$

PROOF. The proof follows on letting $h(t)=\frac{1}{(\log 1/t)^{1+\varepsilon}}$, $y(n)=\frac{1}{\{\log(n+1)\}^{1+\varepsilon}}$ and $\lambda(w)=(\log w)^{1+\varepsilon}$ for $\varepsilon>0$ in the theorem and appealing the second theorem of consistancy for the absolute Riesz summability.

School of Studies in Mathematics,
Vikram University,
UJJAIN-456010, India.

Department of Mathematics,
Madhav Science College,
UJJAIN-456010, India.

REFERENCES

[1] Chandra, P. and Sharma, M.M., *On the absolute Riesz summability of Legendre series*, Indian J. Pure and Applied Math., 10(2) (1979), 191—203.

[2] Chandra, P. and Gupta, R.R., *Absolute summability by Riesz means*, Indian J. Pure and Applied Math. (to appear)

[3] Obrechkoff, N., *Sur la sommation absolue des séries de Dirichlet*, Comptes Rendus, 186(1928), 215—217.

[4] ——————, *Über die absolute summierung der Dirichletschen Reihn*, Math. Zeit., 30(1929), 375—380.

[5] Pal, R.S., and Sharma, M.M., *On the absolute Riesz summability of Legendre series*, The Vikram Univ. J., 21(1977), 79—89.

[6] Sansone, G., *Orthogonal functions*, Interscience Publishers, New York, 1959.

[7] Szegö, G., *Orthogonal polynomials*, Amer. Math. Soc. Colloq. Publications, 1967.