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RECURRENCES FOR THE H-FUNCTION OF TWO VARIABLES

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1. Introduction

The H-function of two variables occuring in this paper is defined in terms of double Mellin Barnes type contour integral in the following manner [7]

$$H[x,y]=H\left[\begin{array}{c|c|c} \begin{pmatrix}0, & n_1\\ p_1, & q_1\end{pmatrix} & \begin{array}{l}(a_j;\ \alpha_j,\ A_j)_{1,p_1}\\ (b_j;\ \beta_j,\ B_j)_{1,q_1}\end{array} & \\ \begin{pmatrix}m_2, & n_2\\ p_2, & q_2\end{pmatrix} & \begin{array}{l}(c_j,\ r_j)_{1,p_2}\\ (d_j,\ \delta_j)_{1,q_2}\end{array} & x, y\\ \begin{pmatrix}m_3, & n_3\\ p_3, & q_3\end{pmatrix} & \begin{array}{l}(e_j,\ E_j)_{1,p_3}\\ (f_j,\ F_j)_{1,q_3}\end{array} & \end{array}\right]$$

$$=\left(\frac{1}{2\pi i}\right)^2 \int_{L_1}\int_{L_2} \phi(s,t)\theta_1(s)\theta_2(t)x^s y^t\, ds\, dt \tag{1.1}$$

where

$$\phi(s,t)=\frac{\prod_{j=1}^{n_1}\Gamma(1-a_j+\alpha_j s+A_j t)}{\prod_{j=1}^{q_1}\Gamma(1-b_j+\beta_j s+B_j t)\prod_{j=n_1+1}^{p_1}\Gamma(a_j-\alpha_j s-A_j t)} \tag{1.2}$$

$$\theta_1(s)=\frac{\prod_{j=1}^{m_2}\Gamma(d_j-\delta_j s)\prod_{j=1}^{n_2}\Gamma(1-c_j+r_j s)}{\prod_{j=m_2+1}^{q_2}\Gamma(1-d_j+\delta_j s)\prod_{j=n_2+1}^{p_2}\Gamma(c_j-r_j s)} \tag{1.3}$$

and with $\theta_2(t)$ defined analogously in terms of the parameters (e_j, E_j), (f_j, F_j). An empty product is interpreted as unity, and all the capitalized parameters are positive. The other conditions on the parameters are analogous to those of the H-function of one variable, and are given in detail by Gupta and Jain [6].

2. The recurrences

For the sake of simplicity and compactness in the recurrences to follow, we write H for the H-function of two variables defined by (1.1) if all its parameters are exactly the same as in (1.1). Also, we write $H[d_1+1]$ for the H-function of two variables in which d_1 is replaced by d_1+1 and all other parameters are the same as in (1.1). Again, for that form of the H-function of two variables in which d_1 is replaced by d_1+1, e_1 by e_1-1 simultaneously, and all other remaining parameters are left unchanged, we use the symbol $H[d_1+1, e_1-1]$, and so on.

RECURRENCE RELATION 1.

$$-\delta_{q_2} F_{q_3} H[c_1-1,\ e_1-1]+r_1E_1 H[d_{q_2}+1,\ f_{q_3}+1]+\delta_{q_2}\begin{vmatrix}(1-e_1) & f_{q_3}\\ -E_1 & F_{q_3}\end{vmatrix}\times$$

$$H[c_1-1]+F_{q_3}\begin{vmatrix}(1-c_1) & d_{q_2}\\ -r_1 & \delta_{q_2}\end{vmatrix} H[e_1-1]=\begin{vmatrix}(1-e_1) & f_{q_3}\\ -E_1 & F_{q_3}\end{vmatrix}\times\begin{vmatrix}(1-c_1) & d_{q_2}\\ -r_1 & \delta_{q_2}\end{vmatrix} H \quad (2.1)$$

provided that $n_i\geq 1,\ q_i>m_i\ (i=2,3)$.

RECURRENCE RELATION 2.

$$\delta_1 E_1 \delta_{q_2} E_{p_3} H[c_{p_2}-1,\ f_{q_3}+1]-F_{q_3} r_{p_2} \delta_1 E_1 H[d_{q_2}+1,\ e_{p_3}-1]$$

$$+r_{p_2} E_1 \delta_{q_2}\begin{vmatrix}f_{q_3} & (e_{q_3}-1)\\ F_{q_3} & E_{p_3}\end{vmatrix} H[d_1+1]+f_{q_3} E_{p_3} \delta_1\begin{vmatrix}d_{q_2} & (c_{p_2}-1)\\ \delta_{q_2} & r_{p_2}\end{vmatrix} H[e_1-1]$$

$$=\begin{vmatrix}(e_{p_3}-1) F_{q_3} E_1 & E_{p_3}\ (c_{p_2}-1)-E_{p_3}\\ \delta_{q_2} f_{q_3} E_1 & r_{p_3} d_{q_3} \quad -\delta_{q_2}\\ F_{q_3} \delta_1(1-e_1) & -r_{p_2} d_1 \quad \delta_1\end{vmatrix} H$$

where

$$m_2\geq 1,\ n_3\geq 1,\ p_i>n_i,\ q_i>m_i\ (i=2,3) \quad (2.2)$$

PROOFS. First of all we note that there are only eleven distinguishable parameters a_1, a_{p_1}, b_1, c_1, c_{p_2}, d_1, d_{q_2}, e_1, e_{p_3}, f_1 and f_{q_3} in the contour integral format for the H-function of two variables defined by (1.1). Next, we observe from the definition of the H-function of two variables, and the application of the formula $\Gamma(z+1)=z\Gamma(z)$, that the replacement of a_1 by a_1-1 in (1.1), is equivalent to the introduction of the additional multiplying factor $(1-a_1+\alpha_1 s+A_1 t)$ into the contour integral format for H. Similarly, the replacement of a_{p_1} by $(a_{p_1}-1)$ introduces $(a_{p_1}-1-\alpha_{p_1}s-A_{p_1}t)$, of b_1 by (b_1+1) introduces $(-b_1+\beta_1 s+B_1 t)$, of c_1 by (c_1-1) introduces $(1-c_1+r_1 s)$, of c_{p_2} by $(c_{p_2}-1)$ introduces $(c_{p_2}-1-r_{p_2}s)$ of d_1 by (d_1+1) introduces $(d_1-\delta_1 s)$, of d_{q_2} by $(d_{q_2}+1)$ introduces $(-d_{q_2}+\delta_{q_2}s)$, of e_1 by (e_1-1) introduces $(1-e_1+e_1 t)$, of e_{p_3} by $(e_{p_3}-1)$ introduces $(e_{p_2}-1-E_{p_2}t)$, of f_1 by (f_1+1) introduces (f_1-F_1t) and of f_{q_3} by $(f_{q_3}+1)$ introduces an additional multiplying factor $(-f_{q_3}+F_{q_3}t)$ into the contour integral format for H. The proof of the recurrences 1 and 2 is based upon the observations mentioned above.

To prove the recurrence relation 1 as an illustration, it is clear from what has been said above that the replacement of c_1 by (c_1-1) and e_1 by (e_1-1) simultaneously will introduce the factor $(1-c_1+r_1s)$ $(1-e_1+E_1t)$: again, the replacement of d_{q_2} by $(d_{q_2}+1)$ and f_{q_3} by $(f_{q_3}+1)$ simultaneously, will introduce the factor $(-d_{q_2}+\delta_{q_2}s)(-f_{q_3}+F_{q_3}\ t)$also, the replacement of c_1 by (c_1-1) alone, will introduce the factor $(1-e_1+E_1t)$. Consequently, we can form the following five term recurrence relation involving indetermined coefficients A,B,C,D and E:

$$AH\,[c_1-1,\ e_1-1]+BH\,[d_{q_2}+1,\ f_{q_3}+1]+CH\,[c_1-1]+DH\,[e_1-1]=EH \quad (2.3)$$

and then require that

$$A(1-c_1+r_1s)(1-e_1+E_1t)+B(-d_{q_2}+\delta_{q_2}s)(-f_{q_3}+F_{q_3}t)+C(1-c_1+r_1s)$$
$$+D(1-e_1+E_1t)=E,$$

be an identity in s,t and st. Hence, A,B,C,D and E can be evaluated. On evaluating the values of these quantities, we easily arrive at the required recurrence relation 1. Recurrence relation 2 can also be obtained in a similar manner.

3. Special cases

(i) If we specialize the parameters of the various H-functions of two variables involved in the recurrence relation 1, such that all of them reduce to Kampé de Fériet function [1], we get, by virtue of a known formula [4], after a little simplification, the following recurrence relation involving Kampé de Fériet functions:

$$-CEF^{0,1}_{0,1}\left[\begin{matrix}-:\ 1+C;\ 1+E\\ -:\quad D;\quad F\end{matrix}\,\middle|\, x,y\right]+(1-D)(1-F)F^{0,1}_{0,1}\left[\begin{matrix}-:C;E\\ -:D-1;\ F-1\end{matrix}\,\middle|\, x,y\right]$$
$$+(E-F+1)\ F^{0,1}_{0,1}\left[\begin{matrix}-:\ 1+C;\ E\\ -:\quad D;\quad F\end{matrix}\,\middle|\, x,y\right]+(C-D+1)E\ F^{0,1}_{0,1}\left[\begin{matrix}-:\ C;\ 1+E\\ -:\ D;\quad F\end{matrix}\,\middle|\, x,y\right]$$
$$=(E-F+1)(C-D+1)\ F^{0,1}_{0,1}\left[\begin{matrix}-:\ C;\ E\\ -:\ D;\ F\end{matrix}\,\middle|\, x,y\right] \quad (3.1)$$

(ii) Again, if we degenerate all the H-function of two variables occurring in the recurrence relation 1, to the Fox's H-function of one variable by means of a known formula [5], we get, after a little change of notation and simplification, the following interesting recurrence relation involving Fox's H-function given earlier by Buschman [2, p. 41(3)]:

$$\beta_q \ H[a_1-1]-\alpha_1 \ H[b_q+1]=-\begin{vmatrix} a_1-1 & b_q \\ \alpha_1 & \beta_q \end{vmatrix} H \tag{3,2}$$

Further, if we degenerate all Fox's H-functions involved in (3.2) into Meijer's G-functions, we arrive at a known result [3, p. 209(11)].

Since, a large number of other special functions of two variables as well as one variable, also follow as special cases of the H-function of two variables as mentioned by Mittal and Gupta [7] and Gupta and Jain [6], corresponding recurrences for these functions can also be obtained easily from our main recurrences merely by suitably specializing the parameters in them. We ommit detail.

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