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A NOTE ON TREE-LIKE SPACES AND THE REGULAR WALLMAN PROPERTY

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A compact Hausdorff space having a regular normal base is called ***regular Wallman.*** In this note we prove that the Stone-Čech compactification of peripherally compact tree-like spaces of weight at most c, are regular Wallman. This is a small improvement of a result of VAN MILL in [2].

The examples of SOLOMON in [7] and ULJANOV in [9] have shown that not all compact Hausdorff spaces are regular Wallman and thus have settled the main question about regular Wallman spaces and Wallman compactifications. But as the class of regular Wallman spaces contains many of the most interesting compact Hausdorff spaces ([2], [3], [4], [8]) a closer insight in them and possible characterizations should still command interest. In these efforts the result of this note is likely to be of interest.

The terminology and notations are of [2] and [4].

1. In a tree-like (or any connected and locally-connected Hausdorff) space segments are open [1] and any two of them, neither of which is a subset of the other, have a non-empty intersection if and only if the boundary point of each is a member of the other. Consequently, if $S_1, S_2, \cdots, S_m$ are segments with distinct boundary points $s_1, s_2, \cdots, s_m$ and such that non is a subset of any other then the intersection of these segments is non-empty if and only if $s_i \in S_j$ for all $i, j, i \neq j$ in $\{1, 2, \cdots, m\}$ and in this case the intersection has the finite boundary $\{s_1, s_2, \cdots, s_m\}$. *Thus, in such a space, any collection of segments such that no two have the same boundary point, is closure-distributive.*

2. In a tree-like space segments separate points and hence in a compact tree-like space the collection of all segments is a subbase for open sets. For compact tree-like spaces of weight at most c, VAN MILL [2] reduced this subbase to one in which no two members have the same boundary point and therefore, in view of the above and ([4], Lemma 3.1), in effect proved that such a space is regular Wallman. As a matter of fact using complements of segments he showed that these spaces are regular supercompact.

3. THEOREM (A). *The Stone-Čech compactification of a peripherally compact, tree-like space of weight at most c is regular Wallman.*

PROOF. Let X be a peripherally compact, tree-like space of weight at most c. Then X has a tree-like compactification whose weight is equal to that of X ([5], Theorem 3). By VAN MILL's theorem referred in §2, this tree-like compactification of X has a closure-distributive subbase for open sets consisting of segments. The trace of this subbase on X, say $\mathbb{S}$, is a closure-distributive subbase for open sets in X. We shall show that the collection $\mathbb{S}^*$ of all sets of the form $A=\bigcup\{A_\alpha : A_\alpha\in\mathscr{A}\}$, $\mathscr{A}\subset\wedge.\mathbb{S}$ and $Cl_X(A)=\bigcup\{Cl_X(A_\alpha) : A_\alpha\in\mathscr{A}\}$, separates closed sets in X.

Let F be a closed subset of X and U any open subset containing F. By ([6], Theorem 1) there exists an open set V such that $F\subset V\subset Cl(V)\subset U$ and the boundary $bd(V)$ of V is discrete. Now as X is collectionwise normal ([6], Theorem 3), for each x in $Cl(V)$ a member $A(x)$ of $\wedge.\mathbb{S}$ can be chosen in such a way that: $x\in A(x)$ and $Cl(A(x))\subset V$ if $x\in V$, $Cl(A(x))\subset U$ if $x\in bd(V)$ and the collection $\{Cl(A(x)) : x\in bd(V)\}$ is discrete. With such a choice, $A=\bigcup\{A(x) : x\in Cl(V)\}=V\cup(\bigcup\{A(x) : x\in bd(V)\})$ is a member of $\mathbb{S}^*$ and $F\subset A\subset Cl(A)\subset U$. Now, an appeal to ([4], Lemma 3.4) completes the proof.

The argument in the above proof is clearly applicable to a more general setting and we may state

THEOREM (B). *Let* X *be a collectionwise normal space in which for any closed subset F and any open set U containing F there is an open set V such that $bd(V)$ is discrete and $F\subset V\subset Cl(V)\subset U$. Then βX is regular Wallman if and only if* X *has a regular Wallman compactification.*

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