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FIXED POINT THEOREMS FOR POINT-TO-POINT AND POINT-TO-SET MAPS IN BANACH SPACE

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1. Introduction

Let $\{x_k\}$ be a sequence in a metric space (M,d). A real nonnegative sequence $\{t_k\}$ is said to majorize $\{x_k\}$ if $d(x_{k+1}, x_k)\leq t_{k+1}-t_k$, $k=0, 1, \cdots$. *If* $\lim_{k\to\infty} t_k = t^* < +\infty$ exists, then $\{x_k\}$ is a Cauchy sequence in M. So if M is complete, there exists a x^* in M such that $\lim_{k\to\infty} x_k = x^*$ and $d(x^*, x_k)\leq t^*-t_k$, $k=0, 1, \cdots$; see [10], [12] and [13]. This observation led to a new proof for the Newton-Kantorovich theorem by Ortega [10] and the general convergence theory for Newton related processes by Rheinboldt [13]. Now suppose that we have the problem of solving a nonlinear equation $F(x)=0$ in a Banach space and that an initial point x_0 for a certain iterative process generates a sequence $\{x_k\}$. Then the problem of finding roots for $F(x)=0$ is reduced to the construction of a convergence scalar sequence $\{t_k\}$ which majorizes $\{x_k\}$. It is our purpose in this paper to formalize fixed point theorems, in the spirit of the above technique, which will have the Banack contraction mapping principle and Browder-Nadler's fixed point theorem as consequences. Among others, some common fixed point theorems are considered.

2. A majorant from of the Browder-Nadler theorem

Throughout this section X will be a Banach Space and M is compact convex subset of Banach space with metric $\| \ \|$.

Let $CL(M)$ be the family of all closed nonempty subsets of M endowed with the generalized Hausdorff metric D induced by $\| \ \|$ [9].

THEOREM 1. *Let* $G: M\to CL(M)$. *Suppose that there exists an isotone function* $\Phi: [0, \infty)\to[0, \infty)$ *such that*

(i) $\Phi(t)<t$ *for each* $t>0$,

(ii) $D(\|G(x)-G(y)\|)\leq\Phi(\min\{\|x-y\|, \|x-G(x)\|, \|x-G(y)\|, \|y-G(x)\|, \|y-G(y)\|\})$, *for each* $x, y\in M$.

Suppose that the sequence $\{x_k\}$ *is defined by iterative process as given* $x_0\in M$,

$p \in (0, 1)$ and $x_k = (1-p)x_{k-1} + pG(x_{k-1})$ for all $n \in N$, and

$$\max\{\|x_{k+1}-x_k\|,\ \|x_{k+1}-G(x_{k+1})\|,\ \|x_{k+1}-G(x_k)\|,\ \|x_k-G(x_{k+1})\|,\ \|x_k-G(x_k)\|,\ \|G(x_{k+1})-G(x_k)\|\} \le \alpha D(\|G(x_k)-G(x_{k-1})\|),\ k=1, 2, \cdots,\ x_0 \in M.$$

Furthermore, let the nonnegative real sequence $\{t_k\}$ be defined by $t_{k+1} = t_k + \Phi(\alpha(t_k - t_{k-1}))$, $t_0 = 0$,

$$t_1 \ge \max\{\|x_1-x_0\|,\ \|x_1-G(x_1)\|,\ \|x_1-G(x_0)\|,\ \|x_0-G(x_1)\|,\ \|x_0-G(x_0)\|,\ \|G(x_1)-G(x_0)\|\},\ k=1, 2, \cdots,$$

converge to $t^* < +\infty$.

Then $\{x_k\}$ converges to a fixed point x^* of G with the error estimate

$$\max\{\|x^*-x_k\|,\ \|x^*-G(x^*)\|,\ \|x^*-G(x_k)\|,\ \|x_k-G(x^*)\|,\ \|x_k-G(x_k)\|,\ \|G(x_k)-G(x^*)\|\} \le \alpha(t^*-t_k),\ k=0, 1, \cdots.$$

PROOF. We show by induction that:

$$\max\{\|x_j-x_{j-1}\|,\ \|x_j-G(x_j)\|,\ \|x_j-G(x_{j-1})\|,\ \|x_{j-1}-G(x_j)\|,\ \|x_{j-1}-G(x_{j-1})\|,\ \|G(x_j)-G(x_{j-1})\|\} \le \alpha(t_j-t_{j-1}),\ j=1, 2, \cdots$$

By assumption for $j=1$ we have:

$$\max\{\|x_1-x_0\|,\ \|x_1-G(x_1)\|,\ \|x_1-G(x_0)\|,\ \|x_0-G(x_1)\|,\ \|x_0-G(x_0)\|,\ \|G(x_1)-G(x_0)\|\} \le (t_j-t_{j-1}),\ j=1, \cdots, k, \text{ then}$$

$$\max\{\|x_{k+1}-x_k\|,\ \|x_{k+1}-G(x_k)\|,\ \|x_{k+1}-G(x_{k+1})\|,\ \|x_k-G(x_{k+1})\|,\ \|x_k-G(x_k)\|,\ \|G(x_{k+1})-G(x_k)\|\} \le \alpha D\|G(x_k)-G(x_{k-1})\|.$$

$$\alpha\Phi(\min\{\|x_k-x_{k-1}\|,\ \|x_k-G(x_k)\|,\ \|x_k-G(x_{k-1})\|,\ \|x_{k-1}-G(x_k)\|,\ \|x_{k-1}-G(x_{k-1})\|,\ \|G(x_k)-G(x_{k-1})\|\} \le \alpha\Phi(\alpha(t_k-t_{k-1})) = \alpha(t_k-t_{k-1}).$$

Since $\lim_{k\to\infty} t_k = t^* < \infty$ exists,

the estimate

(1) $$\max\{\|x_{k+m}-x_k\|,\ \|x_{k+m}-G(x_k)\|,\ \|x_k-G(x_{k+m})\|,\ \|x_k-G(x_k)\|,\ \|G(x_{k+m})-G(x_k)\|\} \le \max \sum_{j=k}^{k+m-1}\|x_{j+1}-x_j\|,\ \sum_{j=k}^{k+m-1}\|x_{j+1}-G(x_j)\|,\ \sum_{j=k}^{k+m-1}\|x_j-G(x_{j+1})\|,\ \sum_{j=k}^{k+m-1}\|x_j-G(x_j)\|,\ \sum_{j=k}^{k+m-1}\|G(x_{j+1})-G(x_j)\|\} \le \sum_{j=k}^{k+m-1}(t_{j+1}-t_j) = (t_{k+m}-t_k),$$

shows that $\{x_k\}$ is a Cauchy sequence. By the completeness of M, there exist $x^* \in M$ such that $\lim_{k\to\infty} x_k = x^*$.

Suppose that $\|x^*-G(x^*)\| = \varepsilon > 0$; and consider

(2) $$\|x_{k+1}-G(x_k)\| \le (1-p)\|x_k-G(x_k)\| + p\|G(x_k)-G(x^*)\| \le (1-p)\|x_k-G(x_k)\| + p\cdot\min\{\|x_k-x^*\|,\ \|x_k-G(x_k)\|,$$

$\|x^*-G(x^*)\|,\ \|x_k-G(x^*)\|,\ \|x^*-G(x_k)\|\}$ and $x_{k+1}-x_k=t(G(x_k)-x_k),\ \lim_{k\to\infty}(G(x_k)-x_k)=0$. Thus $\lim_{k\to\infty}G(x_k)=x^*$.

If $\|x^*-G(x^*)\|>0$ then there is $m\in N$ such that for all $n\geq m,\ n\in N$.

(3) $\|x_k-x^*\|<\min\{x^*-G(x^*)\|,\ \|x^*-G(x_k)\|\}<\|x^*-G(x^*)\|$. It follows from (2) and (3) that for $n\geq m$,

$$\|x_{k+1}-G(x_k)\|\leq(1-p)\|x^*-G(x^*)\|+p\cdot\min\{\|x^*-G(x^*)\|,\ \|x^*-G(x^*)\|,\ \|x^*-G(x^*)\|,\ \|x^*-G(x^*)\|,\ \|x^*-G(x^*)\|\}.$$

Letting $k\to\infty$, we have

$$\begin{aligned}\varepsilon=\|x^*-G(x^*)\|&\leq(1-p)\|x^*-G(x^*)\|+p(\|x^*-G(x^*)\|)\\&<(1-p)\|x^*-G(x^*)\|+p(\|x^*-G(x^*)\|)\\&=\|x^*-G(x^*)\|=\varepsilon,\ \text{a contradiction.}\end{aligned}$$

Hence $x^*=G(x^*)$. The estimate follows from (1) as $m\to\infty$.

COROLLARY 1 (Browder-Nadler [2, 8, 9]). *Let* $G: M\to CL(M)$ *and* $D(\|G(x)-G(y)\|)\leq q\cdot\min\{\|x-y\|,\ \|x-G(x)\|,\ \|x-G(y)\|,\ \|y-G(x)\|,\ \|y-G(y)\|\}$ *for each* x, y *in* M *and* $q\in(0,1)$ *then* G *has a fixed point.*

PROOF. Since $0<q<1$, we have $\frac{1}{q}>1$. Choose $\alpha>1$ in Theorem 1 such that $q\alpha<1$. Let $x_0\in M$.

Choose $x_1=(1-p)x_0+pG(x_0)$ then there exists x_2 such that

$x_2=(1-p)x_1+pG(x_1)$ such that

$$\max\{\|x_2-x_1\|,\ \|x_2-G(x_1)\|,\ \|x_2-G(x_2)\|,\ \|x_1-G(x_1)\|,\ \|x_1-G(x_2)\|,\ \|G(x_2)-G(x_1)\|\}\leq\alpha D(\|G(x_1)-G(x_2)\|).$$

Continuing in this way we produce a sequence $\{x_k\}$ in M such that

$$x_{k+1}=(1-p)x_k+pG(x_k)\ \text{and}$$

$$\max\{\|x_{k+1}-x_k\|,\ \|x_{k+1}-G(x_k)\|,\ \|x_{k+1}-G(x_{k+1})\|,\ \|x_k-G(x_k)\|,\ \|x_k-G(x_{k+1})\|,\ \|G(x_{k+1})-G(x_k)\|\}\leq\alpha D(\|G(x_1)-G(x_2)\|),\ k=1,2,\cdots$$

Let $\Phi(t)=q^t$ in Theorem 1, then $t_{k-1}-t_k=q\alpha(t_k-t_{k-1}),\ t_0=0$,

$$t_1=\max\{\|x_1-x_0\|,\ \|x_1-G(x_1)\|,\ \|x_1-G(x_0)\|,\ \|x_0-G(x_1)\|,\ \|x_0-G(x_0)\|,\ \|G(x_1)-G(x_0)\|\}.$$

So $t_k=\sum_{j=0}^{k-1}(q\alpha)^j\cdot\max\{\|x_1-x_0\|,\ \|x_1-G(x_1)\|,\ \|x_1-G(x_0)\|,\ \|x_0-G(x_1)\|,\ \|x_0-G(x_0)\|,\ \|G(x_1)-G(x_0)\|\}$,

and hence $\lim_{k\to\infty} t_k = t^* = \left[\frac{1}{1-q\alpha}\right]\cdot \max\{\|x_1-x_0\|,\ \|x_1-G(x_1)\|,\ \|x_1-G(x_0)\|,$
$\|x_0-G(x_1)\|,\ \|x_0-G(x_0)\|,\ \|G(x_1)-G(x_0)\|\} < +\infty$

therefore by Theorem 1, G has a fixed point.

If Φ is continuous, then the solution of $t_{k-1}-t_k=\Phi(\alpha(t_k-t_{k-1}))$, $t_0=0$, $k=1$, 2, $\cdots$, satisfies $\lim_{k\to\infty} t_k = t^* < \infty$, so that $\Phi(0)=0$.

Making obvious modifications in Theorem 1, we have:

THEOREM 2. *Let $G: M\to CL(M)$. Suppose that there exists a continuous and isotone function $\Phi: [0,\infty)\to[0,\infty)$ such that*
$D(\|G(x)-G(y)\|)\leq\Phi(\min\{\|x-y\|,\ \|x-G(x)\|,\ \|x-G(y)\|,\ \|y-G(x)\|,\ \|y-G(y)\|\})$
for each x,y in M.

Let $\alpha<1$. Suppose that the sequence $\{x_k\}$ is defined by the iterative process as given $x_0\in M$, $p\in(0,1)$ and $x_k=(1-p)x_{k-1}+pG(x_{k-1})$ for all $k\in N$, and
$\max\{\|x_{k+1}-x_k\|,\ \|x_{k+1}-G(x_{k+1})\|,\ \|x_{k+1}-G(x_k)\|,\ \|x_k-G(x_{k+1})\|,\ \|x_k-G(x_k)\|,$
$\|G(x_{k+1})-G(x_k)\|\}\leq\alpha D(\|G(x_k)-G(x_{k-1})\|)$, $k=1, 2, \cdots$, $x_0\in M$.

Furthermore, let the nonnegative real sequence $\{t_k\}$ be defined by

$$t_{k+1}=t_k+\Phi(\alpha(t_k-t_{k-1})),\ t_0=0,$$

$t_1\geq\max\{\|x_1-x_0\|,\ \|x_1-G(x_1)\|,\ \|x_1-G(x_0)\|,\ \|x_0-G(x_1)\|,\ \|x_0-G(x_0)\|,$
$\|G(x_1)-G(x_0)\|\}$ $k=1,2,\cdots$, *converge to* $t^*<+\infty$.

Then $\{x_k\}$ converges to a fixed point x^* of G with the error estimate
$\max\{\|x^*-x_k\|,\ \|x^*-G(x^*)\|,\ \|x-G(x_k)\|,\ \|x_k-G(x^*)\|,\ \|x_k-G(x_k)\|,$
$\|G(x^*)-G(x_k)\|\}\leq\alpha(t^*-t_k)$, $k=0, 1, \cdots$

THEOREM 3. *Let $C(M)$ be the family of all nonempty compact subsets of M. Let $G: M\to C(M)$. Suppose that there exists an upper right semicontinuous function $\Phi: [0,\infty)\to[0,\infty)$ such that*

(i) $\Phi(t)<t$ *for each* $t>0$

(ii) $D(\|G(x)-G(y)\|)\leq\Phi(\|x-y\|,\ \|x-G(x)\|,\ \|x-G(y)\|,\ \|y-G(x)\|,\ \|y-G(y)\|,$
$\|G(x)-G(y)\|)$

for each x,y in M, then G has a fixed point.

Theorem 3 is a slight generalization of a result of Boyd and Wong [1]. Its proof can be carried over from their proof in [1].

REMARK. Let $\Phi:[0,\infty)\to[0,\infty)$. Suppose that the nonnegative real sequence $\{t_k\}$ satisfies $t_{k+1}-t_k=\Phi(t_k-t_{k-1}), t_0=0,\ t_1$ given, $k=1, 2, \cdots$.

Then the following three conditions, taken together, are not sufficient to imply that $\{t_k\}$ converges.

(a) Φ is isotone and $\Phi(t)<t$ for each $t>0$.

(b) Φ is continuous and isotone.

(c) Φ is upper semicontinuous from the right and $\Phi(t)<t$ for each $t>0$.

In fact, let $\Phi(t)=\frac{t}{1+t}$, $t\in[0,\infty)$. Then Φ satisfies (a), (b) and (c).

Define $\{t_k\}$ by $t_k=\sum_{j=1}^{k}\left(\frac{1}{j}\right)$, $t_0=0$, $j=1, 2, \cdots$. Then $t_{k+1}-t_k=\Phi(t_k-t_{k-1})$, but $\{t_k\}$ is divergent.

THEOREM 4. *Let $\{T_n\}$ be a sequence of (point-to-point) maps from a nonempty Banach space $(M, \|\ \|)$ into itself. Suppose that for each pair (T_i, T_j) there exists a function Φ of $[0,\infty)\times[0,\infty)\times[0,\infty)\times[0,\infty]$ into $[0,\infty)$ such that*

(i) Φ is continuous and isotone in each variable, and $\Phi(t,t,t,t)<t$ for all $t>0$;

(ii) $\|T_i(x)-T_j(y)\|\leq\Phi\cdot\min\{\|x-T_i(x)\|,\ \|y-T_j(y)\|,\ \frac{1}{2}\cdot(\|x-T_j(y)\|+\|y-T_i(x)\|),\ \|x-y\|\}$ for all x, y in M.

Given $x_0\in M$, $p\in(0,1)$ and $x_{k+1}=(1-p)x_k+pT_{k+1}(x_k)$, $k=0, 1, \cdots$, $x_0\in X$.

Assume, further that the sequence $\{t_k\}$ defined $t_{k+1}=t_k+\Phi(t_k-t_{k-1},\ t_k-t_{k-1},\ t_k-t_{k-1},\ t_k-t_{k-1})$ $t_0=0$, $t_1\geq\max\{\|x_1-x_0\|,\ \|x_1-T(x_1)\|,\ \|x_1-T(x_0)\|,\ \|x_0-T(x_1)\|,\ \|x_0-T(x_0)\|,$ $\|T(x_1)-T(x_0)\|\}$. $k=1, 2, \cdots$, converges to $t^*<+\infty$, then $\{x_k\}$ converges to the unique common fixed point x^* of $\{T_n\}$ with error estimate

$\max\{\|x^*-x_k\|,\ \|x^*-T(x^*)\|,\ \|x^*-T(x_k)\|,\ \|x_k-T(x^*)\|,\ \|x_k-T(x_k)\|,$
$\|T(x_k)-T(x^*)\|\}\leq t^*-T_k$, $k=0, 1, \cdots$.

PROOF. We show by induction that $\{t_k\}$ majorizes $\{x_k\}$. By assumption,
$\max\{\|x_1-x_0\|,\ \|x_1-T(x_1)\|,\ \|x_1-T(x_0)\|,\ \|x_0-T(x_1)\|,\ \|x_0-T(x_0)\|,$
$\|T(x_1)-T(x_0)\|\}\leq t_1-t_0$ and if
$\max\{\|x_j-x_{j-1}\|,\ \|x_j-T(x_j)\|,\ \|x_j-T(x_{j-1})\|,\ \|x_{j-1}-T(x_j)\|,\ \|x_{j-1}-T(x_{j-1})\|,$
$\|T(x_j)-T(x_{j-1})\|\}\leq t_j-t_{j-1}$, $j=1, 2, \cdots, k$, then

$$\|x_{k+1}-x_k\|=\|(1-p)x_k+pT_{k+1}(x_k)-(1-p)x_{k-1}-pT_k(x_{k-1})\|$$

$$=(1-p)\|x_k-x_{k-1}+pT_{k+1}(x_k)-T_k(x_{k-1})\|$$
$$<p\cdot\Phi\|T_{k+1}(x_k)-T_k(x_{k-1})\|<\Phi\|x_k-x_{k-1}\|<\|x_k-x_{k-1}\|$$

Therefore,

$$\|x_{k+1}-x_k\|<\Phi\cdot\min\{\|x_k-x_{k-1}\|,\ \|x_k-x_{k-1}\|,\ \|x_k-x_{k-1}\|,\ \|x_k-x_{k-1}\|\}$$
$$\leq\Phi\cdot\min\{(t_k-t_{k-1},\ t_k-t_{k-1},\ t_k-t_{k-1},\ t_k-t_{k-1})\}=t_{k+1}-t_k.$$

Since $\lim_{k\to\infty} t_k=t^*<+\infty$, there exists a x^* in M such that $\lim_{k\to\infty} x_k=x^*$ consider.

(2) $\|x_{k+1}-T_n(x^*)\|\leq(1-p)\|x_k-T_n(x_k)\|+p\|T_n(x_k)-T(x^*)\|$

$$\leq(1-p)\|x_k-T_n(x_k)\|+p\cdot\min\{\|x_k-x^*\|,\ \|x_k-T_n(x_k)\|,$$
$$\|x^*-T_n(x^*)\|,\ \|x_k-T_n(x^*)\|,\ \|x^*-T_n(x_k)\|\}$$

and $x_{k+1}-x_k=t(T_n(x_k)-x_k)$, $\lim_{k\to\infty}(T_n(x_k)-x_k)=0$. Thus $\lim_{k\to\infty} T_n(x_k)=x^*$.

If $\|x^*-T_n(x^*)\|>0$ then there is $m\in N$ such that for all $n\geq m$, $n\in N$.

(3) $\|x_k-x^*\|<\min\{\|x^*-F_n(x^*)\|,\ \|x^*-T_n(x_k)\|\}<\|x^*-F_n(x^*)\|.$

It follows from (2) and (3) that for $n\geq m$.

$$\|x_{k+1}-T_n(x_k)\|\leq(1-p)\|x^*-T_n(x^*)\|+p\cdot\min\ \{\|x^*-T_n(x^*)\|,\ \|x^*-T_n(x^*)\|,$$
$$\|x^*-T_n(x^*)\|,\ \|x^*-T_n(x^*)\|,\ \|x^*-T_n(x^*)\|\}.$$

Letting $k\to\infty$, we have

$$\varepsilon=\|x^*-T_n(x^*)\|\leq(1-p)\|x^*-T_n(x^*)\|+p(\|x^*-T_n(x^*)\|)$$
$$<(1-p)\|x^*-T_n(x^*)\|+p(\|x^*-T_n(x^*)\|)$$
$$=\|x^*-T_n(x^*)\|=\varepsilon,\ \text{a contradiction.}$$

Hence $x^*=T_n(x^*)$ for $n=1, 2, \cdots$. Suppose that $x^*\neq\bar{x}$ and $T_n\bar{x}=\bar{x}$ for each n. Then

$$0<\|x^*-\bar{x}\|$$
$$=\|T_i(x^*)-T_j(\bar{x})\|$$
$$\leq\Phi(0,\ 0,\ \|x^*-\bar{x}\|,\ \|x^*-\bar{x}\|)$$
$$\leq\Phi(\|x^*-\bar{x}\|,\ \|x^*-\bar{x}\|,\ \|x^*-\bar{x}\|,\ \|x^*-\bar{x}\|)$$
$$<\|x^*-\bar{x}\|,\ \text{a contradiction.}$$

Let $T_i=T_j=T$ for each i, j and

$$\Phi(a,b,c,d)=q\cdot\min\ \{a,b,c,d\},\ q<1.$$

Then the results of Kannan [5] and Reich [12] can be easily seen to follow.

3. Selfmaps on a compact convex subset of Banach space

THEOREM 5. *Let* $(M,\|\ \|)$ *be a nonempty compact convex subset of Banach* *space. Let F, G be point-to-set maps of X to $CL(M)$. Suppose that for any distinct x, y in M,*

$D(\|F(x)-G(y)\|)<\min\left\{\frac{1}{2}\|x-F(x)\|+\|y-G(y)\|),\ \frac{1}{2}(\|x-G(y)\|+\|y-G(x)\|),\right.$ $\left.\|x-y\|\right\}$*, then F or G has a fixed point.*

PROOF. Let $\inf\{\|x-F(x)\| : x\in X\}=r_F$ and $\inf\{\|x-G(x)\| : x\in X\}=r_G$. Then there is a sequence $\{x_n\}$ in M such that $\lim_{n\to\infty}\|x_n-F(x_n)\|=r_F$. As a closed subset of a compact convex subset of Banach space. $F(x_n)$ is compact; so there exists u_n in $F(x_n)$such that

$$\|x_n-F(u_n)\|=\|x_n-F(x_n)\|.$$

By the compactness of X, we may, by taking a subsequence, assume that $\{u_n\}$ converges to some $\bar{u}$ in M. If there exist some positive integer p such that $n\geq p$, $x_n=\bar{u}$, then

$$\|\bar{u}-F(\bar{u})\|\leq\inf_{n\geq k}\|\bar{u}-u_n\|,\quad k=p,\ p+1,\ \cdots.$$

So
$$\|\bar{u}-F(\bar{u})\|\leq\sup_{k\geq 1}\inf_{n\geq k}\|\bar{u}-u_n\|=0.$$

Thus $\bar{u}=F(\bar{u})$.

Asssume then that $x_n=\bar{u}$ for infinitely many n's. By taking a subsequence, we may assume then that $x_n\neq\bar{u}$ for each n.

In this case we claim that $\bar{u}$ is a fixed point of G.

Suppose not. Since $u_n=F(x_n)$, by the simple fact that $\|x-A\|\leq\|x-y\|+\|y-A\|$ for $x, y\in M$ and $\Phi\neq A\subset M$, we have

$$\begin{aligned}
\|\bar{u}-G(\bar{u})\| &\leq\|\bar{u}-u_n\|+\|u_n-G(\bar{u})\|\leq\|\bar{u}-u_n\|+\|F(x_n)-G(\bar{u})\| \\
&<\|\bar{u}-u_n\|+\min\left\{\frac{1}{2}(\|x_n-F(x_n)\|+\|\bar{u}-G(\bar{u})\|),\right. \\
&\quad\left.\frac{1}{2}(\|x_n-G(\bar{u})\|+\|\bar{u}\text{-}F(x_n)\|,\ \|x_n-\bar{u}\|\right\} \\
&\leq\|u_n-\bar{u}\|+\min\left\{\frac{1}{2}(x_n-F(x_n)\|+\|\bar{u}-G(\bar{u})\|,\ \frac{1}{2}(\|x_n-u_n\|+\|u_n-\bar{u}\|\right. \\
&\quad\left.+\|\bar{u}-G(\bar{u})\|+\|\bar{u}-F(x_n)\|),\ \|x_n-u_n\|+\|u_n-\bar{u}\|\right\} \\
&=\|\bar{u}-u_n\|+\min\left\{\frac{1}{2}(\|u-F(x_n)\|+\|\bar{u}-G(\bar{u})\|),\ \frac{1}{2}(\|x_n-F(x_n)\|+\|u_n-\bar{u}\|\right. \\
&\quad\left.+\|\bar{u}-G(\bar{u})\|+\|F(x_n)-\bar{u}\|),\ \|x_n-F(x_n)\|+\|u_n-\bar{u}\|\right\}.
\end{aligned}$$

Since $\|F(x_n)-\bar{u}\|\leq\|u_n-\bar{u}\|$ for each n, $\lim_{n\to\infty}\|F(x_n)-\bar{u}\|=0$. Therefore, as $n\to\infty$, we obtain

$$r_G \leq \|\bar{u}-G(\bar{u})\| \leq \min\left\{\frac{1}{2}(r_F+\|\bar{u}-G(\bar{u})\|),\ \frac{1}{2}(r_F+\|\bar{u}-G(\bar{u})\|,\ r_F\right\}.$$

Thus

$r_G \leq \|\bar{u}-G(\bar{u})\| \leq r_F$.

By the same argument we have $r_F \leq r_G$ and hence $r_F=r_G=\|\bar{u}-G(\bar{u})\|$. By the compactness of $G(\bar{u})$, there exist $\bar{u}\neq u^*$. Again, there exists $v^*=F(u^*)$ such that $\|u^*-v^*\| \leq \|G(\bar{u})-F(u^*)\|$ then $\|u^*-v^*\| \leq \|G(\bar{u})-F(u^*)\|$

$$<\min\left\{\frac{1}{2}(\|\bar{u}-G(\bar{u})\|+\|u^*-F(u^*)\|),\right.$$
$$\left.\frac{1}{2}(\|\bar{u}-F(u^*)\|+\|u^*-G(\bar{u})\|),\ \|\bar{u}-u^*\|\right\}$$
$$\leq \min\left\{\frac{1}{2}(\|\bar{u}-G(\bar{u})\|+\|u^*-F(u^*)\|),\right.$$
$$\left.\frac{1}{2}(\|\bar{u}-u^*\|+\|u^*-F(u^*)\|+\|u^*-G(\bar{u})\|,\ \|\bar{u}-u^*\|\right\}$$
$$\leq \min\left\{\frac{1}{2}(r_F+\|u^*-v^*\|,\ \frac{1}{2}(r_F+\|u^*-v^*\|,\ r_F\right\}.$$

Since $\|\bar{u}-G(\bar{u})\|=r_F$, $\|\bar{u}-G(\bar{u})\|=\|\bar{u}-u^*\|$ and $v^*=F(u^*)$. Thus $r_F \leq \|u^*-F(u^*)\| \leq \|u^*-v^*\| < r_F$, a contradiction.

So $\bar{u}=G(\bar{u})$. Therefore F and G has a fixed point. Theorem 5 improves the results in [3, 4, 6, 14, 15] and extends the cases.

$$\alpha_1(t)+\alpha_2(t)+\alpha_3(t)+\alpha_4(t)+\alpha_5(t)=t$$

in [17, Theorem 1 and 2] and

$$\alpha_1(x,y)+\alpha_2(x,y)+\alpha_3(x,y)+\alpha_4(x,y)+\alpha_5(x,y)=1$$

in [18, Theorem 1 and 2]. As an illustration, we give a corollary.

COROLLARY 2. *Let S, T be (point-to-point) self-maps on a nonempty compact convex subset of Banach space $(M, \|\ \|)$. Suppose that there exist functions $\alpha_1=\alpha_2$, $\alpha_3=\alpha_4$, α_5 from $(0,\infty)$ into $[0,\infty)$ such that*

(a) $\alpha_1(t)+\alpha_2(t)+\alpha_3(t)+\alpha_4(t)+\alpha_5(t)\leq t,\ t>0$;

(b) *for any distinct x, y in M,*

$\|S(x)-T(y)\| < a_1\|x-S(x)\|+a_2\|y-T(y)\|+a_3\|x-T(y)\|+a_4\|y-S(x)\|+a_5\|x-y\|$,

where $a^i=\dfrac{\alpha_i(\|x-y\|)}{\|x-y\|}$, $i=1, 2, \cdots, 5$, *then S or T has fixed point. If both S and T have fixed points, then each of S and T has a unique fixed point and* *these fixed points coincide.*

PROOF. Let x, y be distinct points in M. Since $\alpha_1=\alpha_2$, $\alpha_3=\alpha_4$, we have

$$\|S(x)-T(y)\|<\frac{a_1+a_2}{2}(\|x-S(x)\|+\|y-T(y)\|)+\frac{a_3+a_4}{2}(\|x-T(y)\|+\|y-S(x)\|)$$
$$+a_5\|x-y\|.$$
$$\leq\min\left\{\frac{1}{2}(\|x-S(x)\|+\|y-T(y)\|,\ \frac{1}{2}(\|x-T(y)\|+\|y-S(x)\|),\right.$$
$$\left.\|x-y\|\right\}$$

By Theorem 1, S or T has a fixed point.

Moreover, suppose that $\bar{x}$ is a fixed point of S and x^* is a fixed point of T. Then $x=x^*$, since otherwise

$$0<\|\bar{x}-x^*\|=\|S(\bar{x})-T(x^*)\|<\|\bar{x}-x^*\|,\ \text{a contradiction.}$$

(4) Let $x_{2n+1}=(1-p)x_{2n}+pS(x_{2n})$, $x_{2n+2}=(1-p)x_{2n+1}+pT(x_{2n+1})$, $n=0, 1, \cdots$, $x_0\in M$ and $p\in(0,1)$.

THEOREM 6. *Let S, T be (point-to-point) self-maps on a nonempty compact convex subset of Banach space $(M, \|\ \|)$. Suppose that for any distinct x, y in M,*

(5) $\|S(x)-T(y)\|<\min\left\{\frac{1}{2}(\|x-S(x)\|+\|y-T(y)\|),\ \frac{1}{2}(\|x-T(y)\|+\|y-S(x)\|),\right.$ $\left.\|x-y\|\right\}$. *Suppose that S and T are continuous. If S, T have a common fixed point x^*, then the iterative procedure (4) converges to x^* for any initial point x_0 in M.*

PROOF. If $x\neq x^*$, then

$$\|S(x)-x^*\|=\|S(x)-T(x^*)\|$$
$$<\min\left\{\frac{1}{2}\|x-S(x)\|,\ \frac{1}{2}(\|x-x^*\|+\|x^*-S(x)\|),\ \|x-x^*\|\right\}$$
$$<\min\left\{\frac{1}{2}(\|x-x^*\|+\|S(x)-x^*\|),\ \|x-x^*\|\right\}.$$

So $\|S(x)-x^*\|<\|x-x^*\|$ for $x\neq x^*$.

Similarly,

(6) $\|T(x)-x^*\|<\|x-x^*\|$ for $x\neq x^*$. By the compactness of M, there exists a subsequence $\{x_{j(n)}\}$ converging to $\bar{x}$ in M.

If, for some positive integer k, $x_k=x^*$, then the result follows.

For example let $k=2p$; then

$$x_{2n+1}=(1-p)x_{2n}+pS(x_{2n})$$
$$=(1-p)x^*+pS(x^*)$$

$$=x^*-px^*+px^*$$
$$=x^*$$

and hence $x_{2n+2}=(1-p)x_{2n+1}+pT(x_{2n+1})$

$$=(1-p)x^*+pT(x^*)$$
$$=x^*-px^*+px^*$$
$$=x^*.$$

Assume that for each k, $x_k \neq x^*$.

Let $b_n=\|x_n-x^*\|$. Then, by (6), $\{b_n\}$ is decreasing and therefore converges to some number b in $[0,\infty)$. Thus every subsequence, say $\{b_{j(n)}\}$ and $\{b_{j(n)+1}\}$, converges and has the same limit. If $b>0$, then by the continuity of S and T. $0<\|\bar{x}-x^*\|=\lim_{j\to\infty} b_{j(n)}=\lim_{n\to\infty} b_n=\lim_{j\to\infty} b_{j(n)+1}=\|S(\bar{x})-x^*\|$ or $\|T(\bar{x})-x^*\|$, a contradiction to (6).

THEOREM 7. *Let S, T be (point-to-point) self-maps on a nonempty compact convex subset of Banach space $(M, \| \|)$. Suppose that for each proper closed subset K of M, $x,y\in K$, $x\neq y$, (S,T) satisfies (5). Suppose that S and T are continuous. If S, T have a common fixed point x^*, then the iterative procedure (4) converges to x^* for any initial point x_0 in M.*

PROOF. For any $x_0\in M$, define $A(x_0)=\{x : \|x-x^*\|\leq\|x_0-x^*\|\}$. If $x\in A(x_0)$ and $x\neq x^*$, then by (6) $\|S(x)-x^*\|<\|x-x^*\|\leq\|x_0-x^*\|$ and $\|T(x)-x^*\|<\|x-x^*\| \leq\|x_0-x^*\|$. So S and T are self-maps of $A(x_0)$.

Since $A(x_0)$ is compact, by Theorem 2, the iterative procedure (4) converges to x^* for any x_0 in M.

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