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SOME CHARACTERIZATIONS OF SPACES BY EMBEDDINGS IN βX

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1. Introduction

In [2] Arhangelskii defined a space by imposing a certain conditions on a space X in terms of the way it is embedded in its Stone-Čech compactification βX and in [6] R.F. Gittings considered several additional embeddings of a similar nature and showed that these embeddings lead to characterizations of some important classes of spaces.

In this paper, we show that a completely regular space X is metrizable if and only if it is strictly $G_\delta(1)$-embedded in βX by a point normal sequence.

We define O^*-semimetrizable spaces and obtain its some characterizations. Also we obtain that completely regular spaces are metrizable if and only if they are O^*-semimetrizable.

Unless otherwise stated, no separation axioms are assumed; however, topological spaces of Theorem 2.8, 2.11, 2.13 and Corollary 2.9 given in this paper are always T_1-completely regular spaces. The set of positive integers is denoted by N.

2. Definitions and characterizations

If $\mathscr{U}$ is a collection of subsets of a space X and $x\in X$, we define $\mathrm{St}^k(x,\mathscr{U})$ as follows:

$$\mathrm{St}^1(x,\mathscr{U})=\mathrm{St}(x,\mathscr{U})=\bigcup\{U\in\mathscr{U} : x\in U\},$$

$$\mathrm{St}^k(x,\mathscr{U})=\bigcup\{U\in\mathscr{U} : U\cap\mathrm{St}^{k-1}(x,\mathscr{U})\neq\phi\} \text{ for } k\geq 2.$$

The collection $\{\mathrm{St}(x,\mathscr{U}) : x\in U\}$ will be denoted by $\mathscr{U}^*$.

If $\mathscr{U}$ and $\mathscr{V}$ are covers of a space X, we write $\mathscr{U}<\mathscr{V}$ if for every $U\in\mathscr{U}$, there exists $V\in\mathscr{V}$ such that $U\subset V$.

If $\langle\mathscr{U}_n\rangle$ is a sequence of collections of subsets of X with $\mathscr{U}^*_{n+1}<\mathscr{U}_n$ for every $n\in N$, then the sequence $\langle\mathscr{U}_n\rangle$ is called a point normal sequence.

Let X be a completely regular space and let $\langle\mathscr{B}_n\rangle$ be a refining sequence of covers of X by sets open in βX. Consider the following conditions on the sequence $\langle\mathscr{B}_n\rangle$:

(A_k) $\bigcap_{n=1}^{\infty} \mathrm{St}^k(x, \mathscr{B}_n) \subset X$, for each $x \in X$.

(B_k) $\bigcap_{n=1}^{\infty} \mathrm{St}^k(x, \mathscr{B}_n) = \{x\}$ for each $x \in X$.

(C_k) For each $x \in X$ and $n \in N$, there exists $n(x) \in N$ such that

$$\mathrm{Cl}_{\beta X} \mathrm{St}^k(x, \mathscr{B}_{n(x)}) \subset \mathrm{St}^k(x, \mathscr{B}_n).$$

(D_k) For each $x \in X$ and $n \in N$, there exists $n(x) \in N$ such that

$$\mathrm{Cl}_{\beta X} \mathrm{St}^{k+1}(x, \mathscr{B}_{n(x)}) \subset \mathrm{St}^k(x, \mathscr{B}_n).$$

If the completely regular space X has a sequence $\langle \mathscr{B}_n \rangle$ satisfying (A_k) or (B_k), then X is said to be P_k-embedded in βX or $G_\delta(k)$-embedded in βX, respectively. If in addition the sequence $\langle \mathscr{B}_n \rangle$ satisfies (C_k), then, X is said to be strictly P_k-embedded in βX or strictly $G_\delta(k)$-embedded in βX, respectively.

The following Lemmas will be used throughout this paper.

LEMMA 2.1. [6] *Let $\langle \mathscr{B}_n \rangle$ be a sequence of open collections of subsets of βX. If we put $\mathscr{U}_n = \{B \cap X : B \in \mathscr{B}_n\}$ for each $n \in N$, then*

$$\mathrm{St}^k(x, \mathscr{U}_n) = \mathrm{St}^k(x, \mathscr{B}_n) \cap X \text{ for each } x \in X.$$

LEMMA 2.2. [6] *Let $\langle \mathscr{U}_n \rangle$ be a sequence of open covers of X. If we put $\mathscr{B}_n = \{B \text{ open in } \beta X : B \cap X \in \mathscr{U}_n\}$ for each $n \in N$, then*

$$\mathrm{St}^k(x, \mathscr{U}_n) = \mathrm{St}^k(x, \mathscr{B}_n) \cap X \text{ for each } x \in X.$$

LEMMA 2.3. [6] *Let $\langle \mathscr{U}_n \rangle$ be a sequence of open covers of X with $\mathscr{U}_{n+1} < \mathscr{U}_n$ for each $n \in N$. If we put $\mathscr{B}_n = \{B \text{ open in } \beta X : B \cap X \in \mathscr{U}_n\}$, then $\mathscr{B}_{n+1} < \mathscr{B}_n$ for each $n \in N$.*

By the above Lemmas, the following Lemmas are obtained.

LEMMA 2.4. *Let $\langle \mathscr{U}_n \rangle$ be a sequence of open covers of X with $\mathscr{U}^*_{n+1} < \mathscr{U}_n$ for each $n \in N$. If we put $\mathscr{B}_n = \{B \text{ open in } \beta X : B \cap X \in \mathscr{U}_n\}$ for each $n \in N$, then $\mathscr{B}^*_{n+1} < \mathscr{B}_n$ for each $n \in N$.*

LEMMA 2.5. *Let $\langle \mathscr{B}_n \rangle$ be a sequence of covers of X by sets open in βX with $\mathscr{B}^*_{n+1} < \mathscr{B}_n$ for each $n \in N$. If we put $\mathscr{U}_n = \{B \cap X : B \in \mathscr{B}_n\}$ for each $n \in N$, then*

$\mathscr{U}^*_{n+1} < \mathscr{U}_n$ *for each* $n \in N$.

DEFINITION 2.6. A regular space X is called a *Moore space* if there exists a sequence $\langle \mathscr{U}_n \rangle$ of open covers of X such that $\{\mathrm{St}(x, \mathscr{U}_n) : n \in N\}$ is a neighborhood basis at $x \in X$. The sequence $\langle \mathscr{U}_n \rangle$ is called a development for X.

DEFINITION 2.7. A space X is *O-semimetrizable* if there exists a real valued function d on $X \times X$ such that

(1) $d(x, y) = d(y, x) \geq 0$;

(2) $d(x, y) = 0$ if and only if $x = y$;

(3) for $M \subset X$, $x \in \mathrm{Cl}_X M$ if and only if $d(x, M) = \inf\{d(x, y) : y \in M\} = 0$;

(4) for every $\varepsilon > 0$ and $x \in X$, $S_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$ is an open subset of X.

If, in addition, d satifies

(5) for every $\varepsilon > 0$ and $x \in X$, diam $S_d(x, \varepsilon) \leq 2\varepsilon$, then X is said to be *O^*-semimetrizable.*

In Definition 2.7, we don't assume that X is completely regular. A semi-development (1) for a space X is a sequence $\langle \mathscr{U}_n \rangle$ of (not necessarily open) covers of X such that, for every $x \in X$, $\{\mathrm{St}(x, \mathscr{U}_n) : n \in N\}$ is a local system of neighborhoods at x.

THEOREM 2.8. *A space X has a point normal semi-development such that for each $x \in X$ and $n \in N$ there exists $U_n \in \mathscr{U}_n$ such that* $\mathrm{St}(x, \mathscr{U}_{n+1}) \subset \mathrm{Int}_X U_n$ *if and only if it is strictly G_δ* (1)*-embedded in βX by a point normal sequence $\langle \mathscr{B}_n \rangle$ in βX.*

PROOF. *Necessity*; If we put $\mathscr{U}^\circ_n = \{\mathrm{Int}_x U_n : U_n \in \mathscr{U}_n\}$ for each $n \in N$, then $\langle \mathscr{U}^\circ_n \rangle$ is a point normal sequence and $\{\mathrm{St}(x, \mathscr{U}^\circ_n) : n \in N\}$ is a neighborhood basis at $x \in X$.

Now, we put $\mathscr{B}_n = \{B \text{ open in } \beta X : B \cap X \in \mathscr{U}^\circ_n\}$, then by Lemma 2.1 and Lemma 2.4, $\bigcap_{n=1}^{\infty} \mathrm{St}(x, \mathscr{B}_n) \cap X = \bigcap_{n=1}^{\infty} \mathrm{St}(x, \mathscr{U}^\circ_n) = \{x\}$.

Sufficiency; If we put $\mathscr{U}_n = \{B \cap X : B \in \mathscr{B}_n\}$ for each $n \in N$. Then by Lemma 2.5, $\langle \mathscr{U}_n \rangle$ is a point normal sequence of covers of X by sets open in βX. Let, $x \in X$, U be an open set in X containing x and W be an open set in βX such that $W\cap X=U$. Consider the set $H_n=\mathrm{Cl}_{\beta X}\mathrm{St}(x,\mathscr{B}_n)-W$, then there is an $n\in N$ such that $H_n=\phi$. Therefore by Lemma 2.1, we have $\mathrm{St}(x,\mathscr{U}_n)=\mathrm{St}(x,\mathscr{B}_n)\cap X\subset W\cap X=U$, and the theorem is proved.

COROLLARY 2.9. *A space X has a point normal development $\langle\mathscr{U}_n\rangle$ for each $n\in N$ if and only if it is strictly G_δ (1)-embedded in βX by a point normal sequence.*

PROOF. The straightforward proof of this corollary is omitted.

The characterization of metrizable spaces given in Theorem 2.11 is obtained by using the following extension of a metrization theorem of K. Morita [7].

THEOREM 2.10. [7] *For a T_0-space X, the following are equivalent:*

(1) *X is metrizable.*

(2) *There is a sequence $\langle\mathscr{U}_n\rangle$ of open covers of X such that for each $x\in X$ $\{\mathrm{St}^2(x,\mathscr{U}_n) : n\in N\}$ is a neighborhood basis at x.*

THEOREM 2.11. *A space X is metrizable if and only if it is strictly G_δ (1)-embedded in βX by a point normal sequence.*

PROOF. Suppose X is metrizable with a metric d for X and we put $\mathscr{U}_n=\{S_d(x,2^{-n}) : x\in X\}$ for each $n\in N$, then $\langle\mathscr{U}_n\rangle$ is a point normal development. By Corollary 2.9, X is strictly G_δ(1)-embedded in βX by a point normal sequence.

Conversely, suppose X is strictly G_δ(1)-embedded in βX by a point normal sequence $\langle\mathscr{U}_n\rangle$. If we put $\mathscr{U}_n=\{B\cap X : B\in\mathscr{B}_n\}$ for each $n\in N$ and let, $x\in X$, U be an open set in X containing x and W be an open set in βX such that $W\cap X=U$. Then, there is an $n\in N$ such that $\mathrm{St}^2(x,\mathscr{B}_n)\subset W$. Thus by Theorem 2.10 and Lemma 2.5, X is metrizable.

THEOREM 2.12. *A T_0-space X is O^*-semimetrizable if and only if there exists a point normal sequence $\langle\mathscr{U}_n\rangle$ of covers of X such that $\{\mathrm{St}(x,\mathscr{U}_n) : n\in N\}$ is an open neighborhood basis at x.*

PROOF. Let $\mathscr{U}_n=\{U : \mathrm{diam}\ U<3^{-n}\}$ for each $n\in N$. Then $\langle\mathscr{U}_n\rangle$ is a point normal development.

Conversely, suppose $\langle\mathscr{U}_n\rangle$ is a point normal sequence of covers of X such that $\{\mathrm{St}(x,\mathscr{U}_n) : n\in N\}$ is an open neighborhood basis at x, where, without loss of generality, $\mathscr{U}_1=\{x\}$.

For $x, y \in X$, let $n(x, y)$ denote the smallest integer n such that there is no element of $\mathscr{U}_n$ containing both x and y.

If no such integer exists let $n(x, y)=\infty$.

Define $d: X \times X \longrightarrow R$ as follows. For $x, y \in X$,

$d(x, y)=2^{-n(x, y)}$, where $2^{-\infty}=0$. Then d is O^*-semimetric.

Thus X is O^*-semimetrizable with O^*-semimetric d.

THEOREM 2.13. *A space X is metrizable if and only if it is O^*-semimetrizable.*

PROOF. Follows from Theorem 2.8, Theorem 2.11 and Theorem 2.12.

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