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ON HOLOMORPHICALLY PROJECTIVE TRANSFORMATION OF HOLOMORPHICALLY PROJECTIVE RECURRENT KAHLER SPACES

By R.S. Sinha and H.C. Lal

1. Introduction

Recently, Ishihara [1][1)] introduced the concept of holomorphically projective transformation (briefly HP-transformation) in complex manifolds. In the present paper, we will study the effect of HP-transformation on the holomorphically projective recurrent Kahler space (briefly, called $HP\text{-}RK_n$ space). The cases of symmetric and recurrent Kahler spaces (briefly written as $S\text{-}K_n$ and $R\text{-}K_n$ spaces) have also been studied in the concluding article.

Let K_n be an n $(=2m>2)$-dimensional Kahler space with real local coordinates $\{x^i\}$[2)], then we have ([8], p.70):

(1.1) (a) $\varphi_j^{\,r}\varphi_r^{\,i}=-\delta_j^i$ (b) $g_{ji}=g_{rs}\varphi_j^{\,r}\varphi_i^{\,s}$ (c) $\nabla_k\,\varphi_j^{\,h}=0$

where ∇_k denotes the operator of covariant differentiation with respect to the Riemannian metric tensor g_{ji}. Evidently, in a Kahler space, we have the following

(1.2) (a) $\varphi_{ji}=-\varphi_{ij}$ (b) $\varphi^{ji}=-\varphi^{ij}$ (c) $g^{ji}=g^{ab}\varphi_a^{\,j}\varphi_b^{\,i}$
(d) $\varphi_h^{\,h}=0$ (e) $\nabla_k g_{ji}=0$,

where $\varphi_{ji}=\varphi_j^{\,r}g_{ri}$, $\varphi^{ji}=\varphi_t^{\,i}g^{tj}$. Let $R_{kji}^{\;\;\;h}$, $R_{ji}=R_{rji}^{\;\;\;r}$, $R=R_{ji}\,g^{ji}$ be the Riemann curvature tensor, Ricci tensor and the scalar curvature of the space respectively, then the following identities [7,8] are valid in a K_n.

(1.3) (a) $R_{ji}=R_{ab}\,\varphi_j^{\,a}\varphi_i^{\,b}$ (b) $S_{ji}+S_{ij}=0$ (c) $S_{ji}=-\frac{1}{2}\varphi^{tr}R_{trji}$

where

(1.3) (d) $S_{ji}=\varphi_j^{\,r}R_{ri}$.

Let $K_n{}^*$ be another Kahler space obtained by the HP-transformation of K_n, then the christoffel symbols of K_n and $K_n{}^*$ are related by the equation [6];

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1) Numbers in square bracket refer to the references at the end of paper.
2) All the latin indices $i,j,k,\cdots$ run from 1 to n.

$$(1.4)\qquad \left\{ {h \atop ji} \right\}^* = \left\{ {h \atop ji} \right\} + \rho_j \delta_i^h + \rho_i \delta_j^h - \tilde{\rho}_j \varphi_i^h - \tilde{\rho}_i \varphi_j^h,$$

where ρ_i is a certain vector field, $\tilde{\rho}_i = \phi_i^r \rho_r$ and the quantities marked with symbol * denote the quantities of K_n^*. From (1.1)(a), (1.2)(a), (c) and the fact $\tilde{\rho}_i = \varphi_i^r \rho_r$, we immediately have

$$(1.5)\ (a)\ \tilde{\rho}_i \rho^i = 0 \qquad (b)\ \tilde{\rho}_i \tilde{\rho}^i = \rho_i \rho^i \qquad (c)\ \rho_i \tilde{\rho}^i = 0,$$

where

$$(1.5)\ (d)\ \tilde{\rho}^i = g^{ij} \tilde{\rho}_j = -\varphi_r^i \rho^r.$$

If ρ_i in (1.4) vanishes, the transformation becomes affine. Under the *HP*-transformation (1.4), as is well known, holomorphically projective curvature tensor (briefly, *HP*-curvature tensor) P_{kji}^h is invariant [5], i.e.,

$$(1.6)\qquad P^{*h}_{kji} = P^h_{kji},$$

where P^h_{kji} is defined as [6]

$(1.7)\quad P^h_{kji} = R^h_{kji} + \frac{1}{n+2}(R_{ki}\delta^h_j - R_{ji}\delta^h_k + S_{ki}\varphi^h_j - S_{ji}\varphi^h_k + 2S_{kj}\varphi^h_i)$, and satisfies the following ([6], p. 79):

$$(1.8)\quad \begin{array}{lll} (a)\ P^h_{kji} = -P^h_{jki} & (b)\ P^h_{kji} + P^h_{jik} + P^h_{ijk} = 0 & \\ (c)\ P^r_{rji} = P^r_{kri} = P^r_{kjr} = 0 & (d)\ P^r_{kji}\varphi^{hr}_r = P^h_{kjr}\varphi^r_i & \\ (e)\ P^h_{rji}\varphi^r_k = P^h_{rki}\varphi^r_j & (f)\ P^t_{rji}\varphi^r_t = 0 & (g)\ P^t_{kjr}\varphi^r_t = 0 \end{array}$$

From (1.1)(a), (1.2)(b), (c), (1.3)(b), (c), (d), (1.7) and (1.8)(a), (c), (d) and (1.8)(f) by a straight forward calculation we have

$$(1.9)\quad \begin{array}{lll} (a)\ P_{kjih}\, g^{kj} = 0 & (b)\ P_{kjih}\, g^{kh} = 0 & (c)\ P_{kjih}\, g^{jh} = 0 \\ (d)\ P_{kjih}\, g^{ih} = 0 & (e)\ P_{kjih}\, g^{ki} = -A_{jh} & (f)\ P_{kjih}\, g^{ji} = A_{kh} \\ (g)\ P_{kjih}\, \varphi^{kj} = -2\varphi^r_i A_{rh} & (h)\ P_{kjih}\, \varphi^{ki} = \varphi^m_h A_{mj} & (i)\ P_{kjih}\, \varphi^{jh} = 0 \\ (j)\ P_{kjih}\, \varphi^{ih} = 0 & (k)\ P_{kjih}\, \varphi^{kh} = 0 & (l)\ P_{kjih}\, \varphi^{ji} = \varphi^m_k A_{mh} \end{array}$$

where

$$(1.10)\ (a)\ P_{kjih} \equiv P^l_{kji}\, g_{lh} \qquad (b)\ A_{kh} \equiv (1/n+2)(n R_{kh} - R g_{kh}).$$

The tensor A_{kh}, in view of (1.2)(a), (c)(d) and (1.3)(a) satisfies

$$(1.11)\ (a)\ A_{kh} = A_{hk} \quad (b)\ A_{rh}\varphi^r_m \varphi^h_n = A_{mn} \quad (c)\ A_{kh}\, g^{kh} = 0 \quad (d)\ A_{kh}\, \varphi^{kh} = 0.$$

A Kahler space satisfying

(1.12) $$\nabla_l\, P^h_{kji}=k_l\, P^h_{kji},\ \ k_l\neq 0$$

has been called projective recurrent Kahler space [3], but we shall call such Kahler spaces holomorphically projective recurrent Kahler space (briefly HP-R K_n space).

2. HP-transformation of HP-RK_n space

Let us assume that K_n and K^*_n, both, are holomorphically projective recurrent spaces, then (1.12) together with

(2.1) $$\nabla^*_l\, P^{*h}_{kji}=k^*_l\, P^{*h}_{kji},\ \ k^*_l\neq 0$$

holds good. In view of (1.6), equation (2.1) takes the form

(2.2) $$\nabla^*_l\, P^h_{kji}=k^*_l\, P^h_{kji}.$$

But, $\nabla^*_l\, P^h_{kji}=\partial_l\, P^h_{kji}+P^m_{kji}\left\{{h \atop ml}\right\}^*-P^h_{mji}\left\{{m \atop kl}\right\}^*-P^h_{kmi}\left\{{m \atop jl}\right\}^*-P^h_{kjm}\left\{{m \atop il}\right\}^*,$

which on substituting from (1.4) and simplyfying with the help of (1.8)(a), (d) and (1.8)(e) becomes

(2.3) $$\nabla^*_l\, P^h_{kji}=\nabla_l\, P^h_{kji}+(\delta^h_l\, P^m_{kji}\,\rho_m-\rho_k\, P^h_{lji}-\rho_i\, P^h_{kjl}-\rho_j\, P^h_{kli} -2\rho_l\, P^h_{kji})-\varphi^h_l\, P^m_{kji}\,\tilde{\rho}_m+\varphi^m_l(\tilde{\rho}_k\, P^h_{mji}+\tilde{\rho}_j\, P^h_{kmi}+\tilde{\rho}_i\, P^h_{kjm}).$$

Now, we assume that

$$\text{(A)}\quad \nabla^*_l\, P^h_{kji}=\nabla_l\, P^h_{kji},$$

then from (2.3) we find

(2.4) $$\delta^h_l\, P^m_{kji}\,\rho_m-\rho_k\, P^h_{lji}-\rho_j\, P^h_{kli}-\rho_i\, P^h_{kjl}-2\rho_l\, P^h_{kji}-\varphi^h_l\, P^m_{kji}\tilde{\rho}_m +\varphi^m_l(\tilde{\rho}_k\, P^h_{mji}+\tilde{\rho}_j\, P^h_{kmi}+\tilde{\rho}_i\, P^h_{kjm})=0.$$

On contracting (2.4) in the indices h and l and using (1.2)(d), (1.8)(a), (c), (f) and (1.8)(g), we find

(2.5) $$P^m_{kji}\ \rho_m=0.$$

From (1.8)(d) and (2.5) we immediately have

(2.6) $$P^m_{kji}\ \tilde{\rho}_m=0.$$

In view of (2.5) and (2.6), equation (2.4) takes the form

(2.7) $$\rho_k\, P^h_{lji}+\rho_j\, P^h_{kli}+\rho_i\, P^h_{kjl}+2\rho_l\, P^h_{kji}-\varphi^m_l(\tilde{\rho}_k\, P^h_{mji}+\tilde{\rho}_j\, P^h_{kmi}+\tilde{\rho}_i\, P^h_{kjm})=0$$

Now, multiplying (2.5) by g^{ji} and using (1.9)(f), (1.10)(a), we find

$$(2.8) \qquad A_{km}\,\rho^{m}=0 \text{ or } A_{mk}\,\rho^{m}=0$$

and on multiplying (2.6) by g^{ji} and using (1.5)(d), (1.9)(f) and (1.10)(a), we get

$$(2.9) \qquad A_{km}\,\tilde{\rho}^{m}=0 \text{ or } A_{mk}\,\tilde{\rho}^{m}=0.$$

On transvecting (2.7) with $g^{li}\,g_{ht}$ and using (1.2)(b),(c), (1.9)(a), (f), (g), (*l*) and (1.10)(a) we have $\rho_i\,A_{kt}-2\varphi_i^m\,A_{mt}\,\tilde{\rho}_k-\tilde{\rho}_i\,\varphi_k^m\,A_{mh}=2\rho^j\,P_{jkit}$. Thus, transvecting (2.7) with $g^{ji}\,\rho^k g_{ht}$ and using (1.2)(c), (1.5)(a),(d), (1.9)(f), (1.10)(a), (2.8), (2.9) together with the preceding equation, we have $(\rho_k\rho^k)\,A_{lt}=0$, which implies either $\rho_k\rho^k=0$, or $A_{lt}=0$. Hence we have

THEOREM 2.1. *If K_n^* be a HP-transform of K_n and condition* (A) *is satisfied, then one of the following must hold*

(i) $\rho_k\,\rho^k=0$, *i.e. HP-transformation becomes affine,*

(ii) $A_{lt}=0$, *i.e. K_n is an Einstein space.*

Now, if $k_l=k^*_l$, in view of (1.12) and (2.2), we find that condition (A) is satisfied and so from the above theorem, we have

THEOREM 2.2. *If a HP-R K_n space is transformed into another HP-R K^*_n space with same recurrence vector by the HP-transformation* (1.4), *then one of the following must hold good* (i) *transformation is affine* (ii) *K_n is an Einstein space.*

Singh ([3], p.215) has established the following theorem,

THEOREM (B). *If a HP-R K_n space is an Einstein space also, then it reduces to a space of constant holomorphic sectional curvature or the recurrence vector is null.*

In view of the theorem (2.2) and the above theorem we have

THEOREM 2.3. *If a HP-R K_n space is transformed to another HP-R K^*_n space with same recurrence vector by a non affine HP-transformation* (1.4), *then K_n is a space of constant holomorphic sectional curvature or the recurrence vector k_l is a null vector.*

3. The case of $k_l \neq k^*_l$

Now, we study the case in which the recurrence vector k_l of HP-R K_n space and the recurrence vector k^*_l of HP-RK^*_n space, where K^*_n is HP-transform of K_n by (1.4), are unequal. Evidently in this case (1.12), (2.1) and (2.3) hold. On multiplying (2.3) by g_{ht} and using (1.6), (1.10)(a), (1.12) and (2.1), we find

$$(3.1)\quad (k^*_l - k_l)\ P_{kjih} = (g_{lh}\rho_m P^{\ \ m}_{kji} - \rho_k P_{ljih} - \rho_j P_{klih} - \rho_i P_{kjlh} - 2\rho_l P_{kjih}) - [\bar{\rho}_m \varphi_{lh} P^{\ \ m}_{kji} - \varphi^{m}_{l}(\bar{\rho}_k P_{mjih} + \bar{\rho}_j P_{kmih} + \bar{P}_i P_{kjmh})],$$

which on contracting by g^{ji} and using (1.9)(f), (1.10)(a) gives

$$(3.2)\quad (k^*_l - k_l) A_{kh} = g_{lh} A_{km}\rho^m - \rho_k A_{lh} - \rho^i P_{klih} - \rho^i P_{kilh} - 2\rho_l A_{kh} - \varphi_{lh} A_{kt}\bar{\rho}^t + \varphi^{m}_{l}\bar{\rho}_k A_{mh} + \varphi^{m}_{l}(\bar{\rho}^i P_{kmih} + \bar{\rho}^i P_{kimh})$$

On transvecting the skew symmetric part of (3.2) in hand k by g^{lh} and using (1.1)(a), (1.2)(b), (c), (d), (1.9)(a), (c), (d), (e), (g), (h), (i), (j), (1.11)(a), (b), (c) and (1.11)(d) we find

$$(3.3)\ (a)\qquad A_{km}\rho^m = 0 \text{ or } A_{mk}\rho^m = 0.$$

In view of (1.5)(d), (1.11)(b) and (3.3)(a) we at once, have

$$(3.3)\ (b)\qquad A_{km}\bar{\rho}^m = 0 \text{ or } A_{mk}\bar{\rho}^m = 0.$$

Thus, in view of (3.3)(a), (b), equation (3.2) reduces into

$$(3.4)\quad (k^*_l - k_l) A_{kh} = -\rho_k A_{lh} - \rho^i P_{klih} - \rho^i P_{kilh} - 2\rho_l A_{kh} + \varphi^{m}_{l}\bar{\rho}_k A_{mh} + \varphi^{m}_{l}(\bar{\rho}^i P_{kmih} + \bar{\rho}^i P_{kimh}).$$

Now, multiplying (3.4) by ρ^k and using (1.5)(a), (1.8)(a), (e), (1.10)(a) and (3.3)(a), we find

$$(3.5)\ (a)\quad \rho^i \rho^k P_{klih} - \bar{\rho}^i \bar{\rho}^k P_{klih} = -(\rho_k \rho^k) A_{lh} + \varphi^{m}_{l} P_{kimh}\bar{\rho}^i \rho^k$$

whereas, if we multiply (3.4) by $\bar{\rho}^k \varphi^{l}_{t}$, use (1.1)(a), (1.5)(b)(c), (1.8)(a)(e), (1.10)(a) and (3.3)(b), after rearranging the terms, we have

$$(3.5)\ (b)\quad \rho^i \rho^k P_{klih} - \bar{\rho}^i \bar{\rho}^k P_{klih} = (\rho_k \rho^k) A_{lh} + \varphi^{m}_{l} P_{kimh}\bar{\rho}^k \rho^i.$$

From (3.5)(a) and (3.5)(b), in view of (1.8)(a), we have

$$(3.5)\ (c)\qquad P_{klih}\rho^i \rho^k = P_{klih}\bar{\rho}^i \bar{\rho}^k.$$

Using (3.5)(c) in (3.5)(a) and then multiplying the obtained equation by φ^l_s, we have, in view of (1.1)(a)

(3.5) (d) $$P_{kish}\tilde{\rho}^i\rho^k=-(\rho_l\rho^l)A_{rh}\varphi^r_s,$$

whereas using (3.5)(c) in (3.5)(b) and proceeding as above, we find

(3.5) (e) $$P_{kish}\rho^i\tilde{\rho}^k=(\rho_l\rho^l)A_{rh}\varphi^r_s.$$

Thus, transvecting (3.4) with ρ^l and using (1.5) (d), (3.3) (a), (b) and (3.5) (c), we have

(3.6) $$(k^*_l-k_l)\rho^l+2\rho_l\rho^l\ A_{kh}=-4P_{klih}\ \rho^l\rho^i.$$

On the other hand, on multiplying (3.1) by $\rho^k\tilde{\rho}^j\rho^i$ and using (1.5) (a), (b), (c), (d), (1.8)(a), (1.10) (a), (3.3) (a), (3.5) (c) and (3.5) (d) we have $(\rho_k\rho^k)$ $[(\rho_i\rho^i)A_{lh}-2P_{ljih}\rho^j\rho^i]=0$, which implies either $\rho_k\rho^k=0$, or

(3.7) $$(\rho_i\rho^i)A_{lh}=2P_{ljih}\rho^j\rho^i.$$

Hence we have

THEOREM 3.1. *If a HP-R K_n space is transformed into another HP-RK^*_n space with different recurrence vector by HP-transformation* (1.4), *then HP-transformation reduces to an affine transformation or equation* (3.7) *holds good.*

We assume that HP-transformation is non affine, so (3.7) holds. Consequently substituting from (3.7) into (3.6) we have $[(k^*_l-k^l)\rho^l+4(\rho_l\rho_l)]\ A_{kh}=0$ which implies either $A_{kh}=0$, i.e. K_n is an Einstein space, or $(k^*_l-k_l)\rho^l+4(\rho_l\rho^l)=0$. Thus we have

THEOREM 3.2. *If a HP-R K_n space is transformed into another HPR-K^*_n space with different recurrence vector by a non-affine HP-transformation*(1.4), *then either K_n is an Einstein space or* $(k^*_l-k_l)\rho^l+4(\rho_l\rho^l)=0$.

We consider the case $(k^*_l-k_l)\rho^l+4(\rho_l\rho^l)\neq 0$, then by Theorem 3.2 *HP-R* K_n space is an Einstein space also and hence by Theorem (B) either K_n is of constant holomorphic sectional curvature or k_l is a null vector. Moreover, in a K_n of constant *HP*-sectional curvature, $P^h_{kji}=0$ which in view of (1.6) gives $P^{*h}_{kji}=0$, i.e K^*_n is also of constant holomorphic sectional curvature ([8], p.266). Thus we have

THEOREM 3.3. *If a HP-R K_n space is transformed into another HP-RK$^*_n$ space with different recurrence vector by a non-affine HP-transformation and* $(k^*_l - k_l)\rho^l + 4(\rho_1\rho^1) \neq 0$, *then either K_n and K^*_n both are space of constant holomorphic sectional curvature or k_l is a null vector.*

Combining Theorems 3.1, 3.2 and 3.3 we have

THEOREM 3.4. *If a HP-R K_n space with recurrence vector k_l is transformed into another HP-R K^*_n space with recurrence vector $k^*_l(\neq k_l)$ by a HP-transformation* (1.4), *then one of the following cases occur.*

(i) *transformation is affine,* (ii) *K_n and K^*_n both are spaces of constant holomorphic sectional curvature,* (iii) *k_l is a null vector* (iv) $(k^*_l - k_l)\rho^l + 4(\rho_l\rho^l) = 0$.

On the other hand on multiplying (3.4) by $\bar{\rho}^l$ and using (1.5) (c), (d), (3.3) (a)and (3.3)(b) we find $(k^*_l - k_l)\ \bar{\rho}^l\ A_{kh} = 0$. Thus we have

THEOREM 3.5. *If a HP-R K_n space is transformed into another HP-RK$^*_n$ space with different recurrence vector by a HP-transformed* (1.4), *then either $A_{kh}=0$ i.e. K_n is an Einstein space or $(k^*_l - k_l)\ \bar{\rho}^l = 0$, i.e. vectors $(k^*_l - k_l)$ and $\bar{\rho}^l$ form a set of mutually orthogonal vectors.*

In view of Theorem (B) and the discussion before the Theorem 3.3, the above theorem yields.

THEOREM 3.6. *If a HP-RK$_n$ space is transformed into another HP-R K^*_n space with different recurrence vector by a HP-transformation* (1.4) *and* $(k^*_l - k_l)\ \bar{\rho}^l \neq 0$, *then, either K_n and K^*_n both are spaces of constant holomorphic sectional curvature, or k_l is a null vector.*

4. *HP*-transformation of a non-Einstein *HP-R* K_n

Till now we discussed the general case of $HP\text{-}RK_n$ space. Now, in the present article we will study the *HP*-transformation of that *HP-R* K_n space which is not Einstein space, i.e., for which $A_{kh} \neq 0$. In such a case by Theorem 3.5, the relation

$$(4.1) \qquad (k^*_l - k_l)\ \bar{\rho}^l = 0$$

holds good. Also for a non-affine *HP*-transformation, generally called proper

HP-transformation, the relation

(4.2) $$(k^*_l-k_l)\,\rho^l+4(\rho_l\rho^l)=0$$

will hold good due to Theorem 3.2. Substituting from (1.12) and (2.2) into (2.3), we find

(4.3) $$(k^*_l-k_l)\,P^h_{kji}=(\delta^h_l P^m_{kji}\rho_m-\rho_k P^h_{lji}-\rho_j P^h_{kli}-\rho_i P^h_{kji}-2\rho_l P^h_{kji})$$
$$-\varphi^h_l P^m_{kji}\,\tilde{\rho}_m+\varphi^m_l(\tilde{\rho}_k P^h_{mji}+\tilde{\rho}_j P^h_{kmi}+\tilde{\rho}_i P^h_{kjm}),$$

which on contracting in the indices h and l and using (1.2)(d), (1.8)(a), (c), (f) and (1.8)(g) gives

(4.4) $$(k^*_h-k_h)\,P^h_{kji}=(n-2)\,P^h_{kji}\,\rho_h.$$

Thus multiplying (4.3) by $(k^*_h-k_h)$ and using (4.4), we get

(n−3) $$[(k^*_l-k_l)\,P^h_{kji}\,\rho_h]=-(n-2)[(\rho_k P^h_{lji}+\rho_j P^h_{kli}+\rho_i P^h_{kjl}+2\rho_l P^h_{ikj})$$
$$-\varphi^m_l(\tilde{\rho}_k P^h_{mji}+\tilde{\rho}_j P^h_{kmi}+\tilde{\rho}_i P^h_{kjm})]\rho_h$$
$$-\varphi^h_l(k^*_h-k_h)\,P^m_{kji}\,\tilde{\rho}_m.$$

Taking the sum of the above equation with the equations obtained by cyclic interchange of l, k and j in the above equation and using (1.8)(a), (b), (d) and (1.8)(e), we find

(n−3) $$[(k^*_l-k_l)\,P^h_{kji}+(k^*_k-k_k)\,P^h_{jli}+(k^*_j-k_j)P^h_{lki}]\rho_h$$
$$=-\tilde{\rho}_m(k^*_h-k_h)\,[\varphi^h_l P^m_{kji}+\varphi^h_k P^m_{jli}+\varphi^h_k P^m_{jli}+\varphi^h_j \rho^m_{lki}].$$

On transvecting (4.5) with g^{li} and using (1.2)(c), (1.8)(d), (1.9)(e), (f), (1.10)(a), (3.3)(a), (b), the facts $\tilde{\rho}_m=\varphi^r_m\,\rho_r$, $P^h_{kji}\,\rho_h=P_{kjih}\,\rho^h$ and $P^m_{kji}\tilde{\rho}_m=P_{kjim}$ $\tilde{\rho}^m$, we find

(4.6) $$(k^*_l-k_l)\,g^{li}P^h_{kji}\,\rho_h=0$$

since $n>2$. On the other hand multiplying (4.5) by φ^{li} and using (1.1)(a), (1.2)(b), (1.8)(d), (1.9)(h), (l), (1.10)(a), (3.3)(a) and (3.3)(b) we find

(4.7) $$(k^*_l-k_l)\,\varphi^{li}P^h_{kji}\,\rho_h=0,$$

since $n>2$. Thus multiplying (4.3) by $g^{li}\,\rho_h$ and using (1.2)(b), (c), (1.5)(d), (1.8)(d), (1.9)(e), (f), (h), (l), (1.10)(a), (3.3)(a), (b) and (4.6) we have

(4.8) (a) $$P_{kjih}\rho^i\,\rho^h=0,$$

which, in view of (1.1)(b), (1.8)(d) and (1.10)(a) yields

(4.8) (b) $$P_{kjih}\bar{\rho}^i\bar{\rho}^h=0.$$

Again, multiplying (4.3) by $\varphi^{li}\rho_h$ and using (1.2)(b), (c), (1.5)(d), (1.8)(d), (1.9)(e), (f), (h), (l), (1.10)(a), (3.3)(a), (b) and (4.7) we have

(4.9) (a) $$P_{kjih}\bar{\rho}^i\rho^h=0,$$

which in view of (1.8)(d) and (1.10)(a) immediately gives

(4.9) (b) $$P_{kjih}\rho^i\bar{\rho}^h=0.$$

Thus transvecting (3.4) with ρ^h and using (1.8)(d), (1.10)(a), (3.3)(a), (4.8)(a) and (4.9)(a) we find

(4.10) $$P_{kilh}\rho^i\rho^h=P_{kilh}\bar{\rho}^i\bar{\rho}^h.$$

Now multiplying (3.1) by $\bar{\rho}^l\rho^k\rho^h$ and using (1.5)(a), (c), (d), (1.8)(a), (1.10)(a), (3.3)(a), (3.5)(d), (4.1) and (4.8)(a) we find $(\rho_k\rho^k)[P_{ljih}\bar{\rho}^l\rho^h+P_{ljih}\rho^l\bar{\rho}^h]=0$ and hence we have

THEOREM 4.1. *If a non Einstein HP-R K_n space with recurrence vector k_l is transformed into another HPR-K^*_n space with recurrence vector $k^*_l(\neq k_l)$ by a proper HP-transformation, then*

(4.11) $$P_{ljih}\bar{\rho}^l\rho^h+P_{ljih}\rho^l\bar{\rho}^h=0$$

holds good.

Now, multiplying (3.1) by $\rho^k\rho^l$ and using (1.2)(c), (1.5)(a), (d), (1.8)(a), (d), (1.10)(a), (3.7) and (4.2), we find

(4.12) $$-(\rho_a\rho^a)[P_{kjih}\rho^k+\frac{1}{2}\bar{\rho}_iA_{js}\varphi^s_h+\frac{1}{2}\rho_iA_{jh}+\bar{\rho}_jA_{rh}\varphi^r_i)]$$
$$=\rho_hP_{kjim}\rho^m\rho^k+\bar{\rho}_hP_{kjim}\rho^k\bar{\rho}^m,$$

from which on transvecting by $\varphi^j_r\varphi^h_i$, using (1.5)(d), (1.8)(a), (d), (e), (1.10)(a), (1.11)(b) and rearranging the terms, we have

$$-(\rho_a\rho^a)[-P_{kjsh}\varphi^s_i\bar{\rho}^k+\frac{1}{2}\bar{\rho}_iA_{js}\varphi^s_h+\frac{1}{2}\rho_iA_{jh}-\rho_jA_{ih}]$$
$$=-\rho_hP_{kjim}\bar{\rho}^m\bar{\rho}^k+\bar{\rho}_hP_{kjim}\bar{\rho}^k\rho^m.$$

Taking the sum of (4.12) and the above equation, using (4.10), (4.11) and noting the fact $\rho_a\rho^a\neq 0$, we obtain

(4.13) $P_{kjih}\rho^k - P_{kjsh}\varphi_i^s\tilde{\rho}^k = -\tilde{\rho}_i A_{js}\varphi_h^s - \rho_i A_{jh} - \tilde{\rho}_j A_{rh}\varphi_i^r + \rho_j A_{ih}$.

On the other hand, multiplying (3.1) by $\rho^l\rho^i$ and using (1.5)(a),(d), (1.8)(a)(e), (3.7), (4.2), (4.8)(a) and (4.9)(b) we find

(4.14) $P_{kjih}\rho^i = \frac{1}{2}(\rho_j A_{kh} - \rho_k A_{jh} + \tilde{\rho}_j A_{mh}\varphi_k^m - \tilde{\rho}_k A_{mh}\varphi_j^m)$

since $\rho_a\rho^a \neq 0$. On contracting (4.14) with $\varphi_l^h\varphi_s^j$ and using (1.8)(d), (1.10)(a) and (1.11)(b), we find $\varphi_s^j P_{kjrt}\tilde{\rho}^r = \frac{1}{2}(\tilde{\rho}_s A_{kh}\varphi_t^h - \rho_k A_{st} - \rho_s A_{kt} - \tilde{\rho}_k A_{jt}\varphi_s^j)$. From (4.14) and the above equation, in view of (1.11)(b), we have

(4.15) $P_{kjih}\rho^i - \varphi_j^m P_{kmih}\tilde{\rho}^i = \rho_j A_{kh} + \tilde{\rho}_j A_{mh}\varphi_k^m$.

Substituting from (4.13) and (4.15) in (3.4) and using (1.8)(a), (1.11)(b), we find that $k^*_l - k_l = -4\rho_l$. Thus we get

THEOREM 4.2. *If a non Einstein HP-R K_n space with recurrence vector k_l is transformed into another HP-RK^*_n space with recurrence vector $k^*_l(\neq k_l)$ by a proper HP-transformation, then $(k^*_l - k_l) = -4\rho_l$.*

5. *HP*-transformation of R-K_n and S-K_n spaces

This article is devoted to the study of *HP*-transformation of R-K_n and S-K_n spaces. Suppose K_n and its *HP*-transform K^*_n both are recurrent spaces with k_l and k^*_l as recurrence vectors respectively. Then

(5.1) $\nabla_l R^h_{kji} = k_l R^h_{kji}$ and $\nabla^*_l R^{*h}_{kji} = k^*_l R^{*h}_{kji}$.

The following theorem is well known.

THEOREM (C) ([4], p.78). *A R-K_n space is a HP-R K_n space with same recurrence vector.*

Consequently from (1.6), (5.1) and the above theorem, we have

(5.2) $\nabla_l P^h_{kji} = k_l P^h_{kji}$ and $\nabla^*_l P^{*h}_{kji} = k^*_l P^h_{kji}$.

Now, if $k_l = k^*_l$ we see that condition (A) of §2 is trivially satisfied and hence from Theorem 2.1, we have

THEOREM 5.1. *If a R-K_n space is transformed into another R-K^*_n space with same recurrence vector by HP-transformation* (1.4) *then either K_n is an Einstein space or the transformation is affine.*

Now, in a Kahler space of constant holomorphic sectional curvature, curvature tensor R^h_{kji} takes the form $R^h_{kji}=\frac{k}{4}[(\delta^h_k g_{ji}-\delta^h_j g_{ki})+(\varphi^h_k \varphi_{ji}-\varphi^h_j \varphi_{ki})-2\varphi_{kj}\varphi^h_i]$ ([8], p. 71), where k is an absolute constant. Differentiating the above equation covariantly with respect to x^l and using (1.1)(c), (1.2)(c) we find $\nabla_l R^h_{kji}=0$ and so if the space be recurrent also, in view of (5.1) we find $R^h_{kji}=0$, since $k_l\neq 0$, i.e., the space under consideration is a flat space. So, we remark

REMARK 5.1. A R-K_n space of constant holomorphic sectional curvature is a flat space.

Thus in view of Theorem 3.4, Theorem 4.2, Theorem (C) and Remark 5.1 we have

THEOREM 5.2. *If a R-K_n space with recurrence vector k_l is transformed into another R-K^*_n space with recurrence vector $k^*_l(\neq k_l)$ by a HP-transformation* (1.4), *then one of the following cases occur:*

(i) *transformation is affine,* (ii) *K_n and K^*_n both are flat spaces*

(iii) *k_l is a null vector,* (iv) $k^*_l-k_l+4\rho_l=0$.

In case K_n and K^*_n both are symmetric spaces, we have $\nabla_l R^h_{kji}=0$ and $\nabla^*_l R^{*h}_{kji}=0$, consequently we can have $\nabla_l P^h_{kji}=0=\nabla^*_l P^{*h}_{kji}$ and hence condition (A) of §2 is identically satisfied. Therefore, from Theorem 2.1 we have

THEOREM 5.3. *If a S-K_n is transformed into another S-K^*_n space by HP-transformation* (1.4) *then, either K_n is an Einstein space, or, the transformation is affine.*

We conclude the article by considering the case when K_n is a R-K_n space with recurrence vector k_l and K^*_n is a HP-R K_n space with k^*_l as recurrence vector, i.e., $\nabla_l R^h_{kji}=k_l R^h_{kji}$ and $\nabla^*_l P^{*h}_{kji}=k^*_l P^{*h}_{kji}$ hold. If $k_l=k^*_l$, in view of Theorem (C) and 1.6 we see that condition (A) of §2 is trivially satisfied. Therefore from Theorem 2.1 we have

THEOREM 5.4. *If a R-K_n space is transformed into a HP-R K^*_n space with same recurrence vector by a HP-transformation* (1.4) *then either K_n is an Einstein space or transformation is affine.*

Moreover, with the help of Theorems 3.4, 4.2, (C) and Remark 5.1 we can have the

THEOREM 5.4. *If a R-K_n space with recurrence vector k_l is transformed into a HP-RK_n space with recurrence vector $k^*_l(\neq k_l)$, by a HP-transformation* (1.4), *then one of the following cases occur*: (i) *transformation is affine*, (ii) K_n *is a flat space*, K^*_n *is a space of constant holomorphic sectional curvature*, (iii) k_l *is a null vector*, (iv) $k^*_l - k_l + 4\rho_l = 0$.

Department of mathematics
University of Gorakhpur,
Gorakhpur 273001, U.P., INDIA.

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