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OPERATIONAL FORMULAE ASSOCIATED WITH A CLASS OF GENERALIZED POLYNOMIALS

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1. Introduction

Recently Gould and Hopper [8] established the operational formula

$$\mathscr{D}^n=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k} H^r_{n-k}(x,\ \alpha,\ p)D^k, \quad \cdots\cdots(1.1)$$

where
$$x^n\mathscr{D}^n=\prod_{j=1}^{n-1}(xD-prx^r+\alpha-j), \quad \cdots\cdots(1.2)$$

for the generalized Hermite polynomials

$$H^r_n(x,\ \alpha,\ p)=(-1)^n x^{-\alpha} e^{px^r} D^n(x^\alpha e^{-px^r}) \quad \cdots\cdots(1.3)$$

The relation (1.1) is a generalization of the operational formula of Burchnall [3] for Hermite polynomials as well as of Carlitz [4] for Laguerre polynomials.

Al-Salam [2], Chatterjea [5, 6, 7], Munot and Saxena [10], Singh [12] and many others have also obtained the operational formulae for the classical polynomials and have derived either some new formulae or the known ones by alternative methods.

Employing the operator $T_q\equiv x^{q+1}D$, Joshi and Prajapat [8] defined a class of polynomials $M^{(\alpha)}_{vn}(x\,;q)$ by

$$M^{(\alpha)}_{vn}(x\,;q)=\frac{1}{n!}\,x^{-\alpha-nq}\,e^{p_v(x)}\,T^n_q\,[x^\alpha\,e^{-p_v(x)}], \quad \cdots\cdots(1.4)$$

where $p_v(x)$ is the polynomial in χ of degree v and q is a constant.

In the present paper we have obtained some operational formulae for the polynomial $M^{(\alpha)}_{vn}(x,\ q)$ and have applied them to derive some interesting results.

2. The operational formulae

The following results are required in our investigations:

$$F(\delta)[x^\alpha f(x)]=x^\alpha F(\delta+\alpha)f(x), \quad \cdots\cdots(2.0)$$

$$F(\delta)[\exp(g(x))f(x)]=\exp(g(x))F(\delta+xg')f(x), \quad \cdots\cdots(2.1)$$

$$x^{-qn}\,T^n_q=\delta(\delta+q)\cdots(\delta+q(n-1)), \quad \cdots\cdots(2.2)$$

$$T^n_q(x^\alpha)=q^n\left(\tfrac{\alpha}{q}\right)_n x^{\alpha+qn}, \qquad \cdots(2.3)$$

$$F(T_q)[x^\alpha f(x)]=x^\alpha F(T_q+\alpha x^q)f(x), \qquad \cdots(2.4)$$

$$F(T_q)[\exp(g(x))f(x)]=\exp(g(x))F(T_q+x^{q+1}g')f(x), \qquad \cdots(2.5)$$

$$T^n_q(uv)=\sum_{m=0}^{n}\binom{n}{m}T^{n-m}_q(u)T^m_q(v), \qquad \cdots(2,6)$$

where $\delta\equiv xD$.

The application of the results (2.0), (2.1) and (2.2), provides us with

$$T^n_q(x^\alpha e^{-p_v(x)}Y)=e^{-p_v(x)}x^{\alpha+qn}\prod_{j=1}^{n}(\delta+\alpha+(n-j)q-xp_v'(x))Y \qquad \cdots(2.7)$$

where Y is a sufficiently differentiable function of x and $p_v'(x)\equiv\frac{d}{dx}p_v(x)$, whereas the formulae (2.4) and (2.5) leads us to

$$T^n_q[x^\alpha e^{-p_v(x)}Y]=x^{\alpha+(q+1)n}e^{-p_v(x)}\left[D+\frac{\alpha}{x}-p'_v(x)\right]^nY. \qquad \cdots(2.8)$$

It is also easy to see that

$$T^n_q[x^\alpha e^{-p_v(x)}Y]=\sum_{m=0}^{n}\binom{n}{m}(n-m)!\,x^{\alpha+q(n-m)}e^{-p_v(x)}M^{(\alpha)}_{v(n-m)}(x,q)T^m_qY \cdots(2.9)$$

Thus comparison of (2.7) and (2.9) yields the operational formula

$$\prod_{j=1}^{n}(\delta+\alpha+(n-j)q-xp_v'(x))Y=\sum_{m=0}^{n}\frac{n!}{m!}x^{-qm}M^{(\alpha)}_{v(n-m)}(x,q)T^m_qY, \qquad \cdots(2.10)$$

and comparison of (2.8) and (2.9) results into an another operational formula

$$\left(D+\frac{\alpha}{x}-p_v'(x)\right)^n.\ Y=\sum_{m=0}^{n}\frac{n!}{m!}x^{-(qm+n)}M^{(\alpha)}_{v(n-m)}(x,q)\,T^m_qY \qquad \cdots(2.11)$$

For $Y=1$ in (2.11), we have

$$\left(D+\frac{\alpha}{x}-p_v'(x)\right)^n.\ 1=n!\ x^{-n}M^{(\alpha)}_{vn}(x,q). \qquad \cdots(2.12)$$

Next, using (2.6), we have

$$T^n_q[x^\alpha e^{-p_v(x)}Y]=\sum_{s=0}^{n}\binom{n}{s}T^s_q(x^\alpha)T^{n-s}_q(e^{-p_v(x)}Y),$$

which on using (2.3) and (2.5) gives

$$T^n_q[x^\alpha e^{-p_v(x)}Y]=\sum_{s=0}^{n}\binom{n}{s}q^s\left(\frac{\alpha}{q}\right)_s x^{\alpha+qs}e^{-p_v(x)}(T_q-x^{q+1}p_v'(x))^{n-s}Y$$

$$=\sum_{m=0}^{n}\binom{n}{n-m}q^{n-m}\left(\frac{\alpha}{q}\right)_{n-m}x^{\alpha+q(n-m)}e^{-p_v(x)}(T_q-x^{q+1}p_v'(x))^mY$$

$$=\sum_{m=0}^{n}\lambda(m,n)e^{-p_v(x)}x^{\alpha+q(n-m)}(T_q-x^{q+1}p_v'(x))^mY$$

where $\lambda(m,n)=\binom{n}{n-m}q^{n-m}\left(\frac{\alpha}{q}\right)_{n-m}$, which in view of (2.9) yields the formula

$$\sum_{m=0}^{n}\lambda(m,n)x^{m}(D-p_v'(x))^{m}Y=\sum_{r=0}^{n}\frac{n\,!}{r\,!}x^{-qr}M_{v(n-r)}^{(\alpha)}(x,q)T_q^{r}Y. \quad \cdots\cdots\cdots(2.13)$$

3. Particular cases

In an special case when $\alpha=0$, $p_v(x)=x^2$, $q=-1$, and $M_{vn}^{(0)}(x,-1)=\frac{(-x)^n}{n\,!}$ $H_n(x)$, the formula (2.10) yields

$$\prod_{j=1}^{n}(\delta-(n-j)-2x^2)Y=\sum_{m=0}^{n}\binom{n}{m}x^{n}(-1)^{n-m}H_{n-m}(x)D^{m}Y, \quad \cdots\cdots\cdots\cdots(3.1)$$

Which can also be written as

$$\prod_{j=1}^{n}(\delta-j+1-2x^2)Y=\sum_{m=0}^{n}\binom{n}{m}x^{n}(-1)^{n-m}H_{n-m}(x)D^{m}Y, \quad \cdots\cdots\cdots\cdots\cdots(3.2)$$

and when $Y=1$, it reduces to

$$\prod_{j=1}^{n}(\delta-2x^2-j+1)\cdot 1=(-x)^{n}H_n(x), \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(3.3)$$

a formula proved earlier by Al-Salam [1] by a different method.

On taking $q=-1$, $\alpha=0$, $p_v(x)=x^2$ and $M_{vn}^{(0)}(x,-1)=\frac{(-x)^n}{n\,!}H_n(x)$, the formula (2.11) becomes,

$$(D-2x)^{n}=\sum_{m=0}^{n}\binom{n}{m}(-1)^{n-m}H_{n-m}(x)D^{m}, \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots(3.4)$$

which was proved earlier by Burchnall [3].

Further on taking $q=-1$, $p_v(x)=px^r$ and $M_{vn}^{(\alpha)}(x,-1)=\frac{(-x)^n}{n\,!}H_n^{r}(x,\alpha,p)$, the formula (2.12) gives

$$\left(D+\frac{\alpha}{x}-prx^{r-1}\right)^{n}\cdot 1=(-1)^{n}H_n^{r}(x,\alpha,p), \quad \cdots\cdots\cdots\cdots\cdots\cdots(3.5)$$

which is given earlier by Gould and Hopper [8].

Next, on replacing α by $\alpha+m$ and taking $p_v(x)=x$, $q=-1$ and $M_{vn}^{(\alpha+n)}(x,-1)$ $=L_n^{(\alpha)}(x)$, the formula (2.13) yields

$$\sum_{m=0}^{n}\binom{\alpha+n}{n-m}\frac{x^{m}}{m!}(D-1)^{m}Y=\sum_{r=0}^{n}\frac{x^{r}}{r\,!}L_{n-r}^{(\alpha+r)}(x)\,D^{r}Y, \quad \cdots\cdots\cdots\cdots\cdots(3.6)$$

given earlier by Chatterjea [5].

Lastly if we set $p_v(x)=px^r$, and $M_{vn}^{(\alpha)}(x,q)=G_n^{(\alpha)}(x,r,p,k)$, where $G_n^{(\alpha)}(x,$

$r, p, k)$ is the polynomial defined by Srivastava and Singhal [13] then the results (2.10), (2.11), (2.13) respectively immediately gets us with

$$\prod_{j=1}^{n}(\delta+\alpha+(n-j)q-rpx^{r})Y=\sum_{s=0}^{n}\frac{n!}{s!}x^{-ks}G_{n-s}^{(\alpha)}(x,r,p,k,)\theta^{s}Y, \qquad \cdots\cdots(3.7)$$

$$\left(D-\frac{\alpha}{x}-rpx^{r-1}\right)^{n}Y=\sum_{m=0}^{n}\frac{n!}{m!}x^{-(km+n)}G_{n-m}^{(\alpha)}(x,r,p,k)\theta^{m}Y, \qquad \cdots\cdots(3.8)$$

and

$$\sum_{m=0}^{n}\lambda(m,n)x^{-km}(\theta-rpx^{r+k})^{m}Y$$

$$=\sum_{s=0}^{n}\frac{n!}{s!}x^{-ks}G_{n-s}^{(\alpha)}(x,r,p,k)\,\theta^{s}Y, \qquad \cdots\cdots(3.9)$$

where $\theta\equiv x^{k+1}D$.

Now using the result

$$Y_{n}^{(\alpha)}(x,k)=-k^{n}\,G_{n}^{(\alpha+1)}(x,1,1,k),\ \alpha>-1,\ k=1,2,\cdots,\ [14],$$

where $Y_{n}^{(\alpha)}(x,k)$ is the Konhauser's biorthogonal polynomials, in the results mentioned above, we can easily deduce the operational formulae for the polynomials $Y_{n}^{(\alpha)}(x;k)$.

4. Some applications

(i) Two generating functions for the polynomial set $\{M_{vn}^{(\alpha)}(x,q)\}$.

Firstly we establish that

$$\sum_{n=0}^{\infty}\binom{m+n}{n}M_{v(m+n)}^{(\alpha-nq)}(x,q)t^{n}=(1+qt)^{\frac{\alpha-q}{q}}\exp\left[p_{v}(x)-p_{v}(x(1+qt)^{1/q})\right]$$

$$\cdot M_{vm}^{(\alpha)}\left[x(1+qt)^{1/q},\ q\right] \qquad \cdots\cdots(4.1)$$

This result is believed to be new.

PROOF. On setting $Y=1$ in (2.10), we have,

$$\prod_{j=1}^{n}(\delta+\alpha+(n-j)q-xp_{v}'(x))\cdot 1=n!\ M_{vn}^{(\alpha)}(x,q) \qquad \cdots\cdots(4.2)$$

Obviously,

$$(m+n)!\ M_{v(m+n)}^{(\alpha-nq)}(x,q)=\prod_{j=1}^{m+n}(\delta+\alpha+(m-j)q-xp_{v}'(x))$$

$$=\prod_{j=1}^{n}(\delta+\alpha-jq-xp_{v}'(x))\prod_{j=1}^{m}(\delta+\alpha+(m-j)q-xp_{v}'(x))$$

$$= \prod_{j=1}^{n} (\delta+\alpha-jq-xp_v'(x)) \cdot m!\ M_{vm}^{(\alpha)}(x, q),$$

which in view of (2.0) and (2.1) gives

$$(m+n)!\ M_{v(m+n)}^{(\alpha-nq)}(x,q)=m!\ x^{-\alpha+q}\, e^{p_v(x)} \cdot (-q)^n \prod_{j=1}^{n}\left(-\frac{\delta}{q}+(j-1)\right) \cdot \{x^{\alpha-q}\, e^{-p_v(x)}\, M_{vm}^{(\alpha)}(x, q)\}.$$

Now, on multiplying both sides by $\frac{t^n}{n!}$ and then summing for n from 0 to ∞, we immediately get

$$\sum_{n=0}^{\infty} \binom{m+n}{n} M_{v(m+n)}^{(\alpha-nq)}(x,q)t^n = x^{-\alpha+q}\, e^{p_v(x)}\, (1+qt)^{\delta/q} \cdot \{x^{\alpha-q}\, e^{-p_v(x)}\, M_{vm}^{(\alpha)}(x,q)\}.$$

On using the result

$$a^{\delta} f(x)=f(ax), \quad \cdots\cdots (4.3)$$

it yields the desired generating function.

For $m=0$, it is easy to see that

$$\sum_{n=0}^{\infty} M_{vn}^{(\alpha-nq)}(x,q)t^n=(1+qt)^{\frac{\alpha-q}{q}} \cdot \exp[p_v(x)-p_v(x(1+qt)^{1/q})], \quad \cdots\cdots (4.4)$$

which for $p_v(x)=px^r$ reduces to the known result by Srivastava and Singhal [13, p. 79].

Secondly we prove that

$$\sum_{m=0}^{\infty} \binom{m+n}{n} M_{v(m+n)}^{(\alpha)}(x,q)t^m=(1-qt)^{-n-\frac{\alpha}{q}} \exp[p_v(x)-p_v(x(1-qt)^{-1/q})] \cdot M_{vn}^{(\alpha)}(x(1-qt)^{-1/q}, q), \quad \cdots\cdots (4.5)$$

proved earlier by Rai and Singh [11] by some different method.

PROOF. Proceeding parallel to the proof of (4.1), we easily obtain the generating function (4.5).

(ii) Recurrence relations:

(a) Firstly we observe that the recurrence relation

$$\left(D+\frac{\alpha}{x}-p_v'(x)\right) M_{vn}^{(\alpha)}(x,q)=(n+1)x^{-1}\, M_{v(n+1)}^{(\alpha)}(x,q), \quad \cdots\cdots (4.6)$$

is an easy consequence of the result (4.2).

Iteration of the process used in the proof of (4,6) gives us the relation

$$\left(D+\frac{\alpha}{x}-p_v'(x)\right)^m M_{vn}^{(\alpha)}(x,q)=(n+1)_m\, x^{-m}\, M_{v(n+m)}^{(\alpha)}(x,q) \cdots\cdots (4.7)$$

On setting $q=-1$, $p_v(x)=px^r$ and $M_{vn}^{(\alpha)}(x,-1)=\frac{(-x)^n}{n!}\ H_n^r(x,\alpha,p)$, in (4.7), we immediately get

$$\mathscr{D}^m H_n^r(x, \alpha, p)=(-1)^m H_{n+m}^r(x, \alpha, p), \qquad \cdots\cdots(4.8)$$

given earlier by Gould and Hopper [8].

(b) Joshi and Prajapat [9] have shown that

$$\sum_{n=0}^{\infty} M_{vn}^{(\alpha)}(x, q)t^n=(1-qt)^{-\alpha/q} \exp[p_v(x)-p_v(x(1-qt)^{-1/q})] \qquad \cdots\cdots(4.9)$$

Using (4.4) and (4.9) we immediately get

$$\sum_{n=0}^{\infty} M_{vn}^{(\alpha)}(x, -q)t^n=(1+qt)\sum_{n=0}^{\infty} M_{vn}^{(\alpha-nq)}(x,q)t^n \qquad \cdots\cdots(4.10)$$

Thus, on equating the coefficients of t^n on two sides of (4,10), we get the pure recurrence relation

$$M_{vn}^{(\alpha)}(x,-q)=M_{vn}^{(\alpha-nq)}(x,q)+q\,M_{v(n-1)}^{(\alpha-nq-q)}(x,q).$$

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