Kyungpook Math. J.
Volume 22, Number 1
June, 1982

AN APPLICATION OF THE FRACTIONAL CALCULUS

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1. Introduction

There are many definitions of the fractional calculus. At first J. Liouville [1] defined the fractional derivative of order α. Then, T. J. Osler defined the fractional derivative of order α in [3]. In 1974, B. Ross defined the fractional derivative of order α in [5]. Moreover, K. Nishimoto defined the fractional derivative and integral of order α in [2]. And in 1978, M. Saigo defined the integral operators in [6]. Furthermore, in 1978, S. Owa gave the following definition for the fractional integral of order α in [4].

DEFINITION 1. The fractional integral of order α is defined by

$$D_z^{-\alpha} f(z)=\frac{1}{\Gamma(\alpha)}\int_0^z \frac{f(\zeta)d\zeta}{(z-\zeta)^{1-\alpha}},$$

where $\alpha>0$, $f(z)$ is an analytic function in a simply connected region of the z-plane containing the origin, and the multiplicity of $(z-\zeta)^{\alpha-1}$ is removed by requiring $\ln(z-\zeta)$ to be real when $(z-\zeta)>0$. Moreover,

$$f(z)=\lim_{\alpha\to 0} D_z^{-\alpha} f(z).$$

Let E be a domain in the extended complex plane. The function $f(z)$ is called *univalent* in E if and only if it is analytic except for at most one pole and $f(z_1)\neq f(z_2)$ for $z_1\in E$, $z_2\in E$, and $z_1\neq z_2$.

DEFINITION 2. Let S_0 denote the class of functions

$$f(z)=|a_1|z-\sum_{n=2}^{\infty}|a_n|z^n$$

that are analytic univalent in the unit disk U. And let $S^*_0(k)$ denote the class of functions

$$f(z)=|a_1|z-\sum_{n=2}^{\infty}|a_n|z^n$$

that are starlike of order k $(0\leqq k<1)$ with respect to the origin in the unit disk U. Furthermore, let $K_0(k)$ denote the class of functions

$$f(z)=|a_1|z-\sum_{n=2}^{\infty}|a_n|z^n$$

that are convex of order k ($0\leqq k<1$) in the unit disk U.

2. Some results for the fractional integral

Now, we have immediately the following lemmas by means of some results were shown by H. Silverman in [7].

LEMMA 1. *A function*

$$f(z)=|a_1|z-\sum_{n=2}^{\infty}|a_n|z^n$$

is in the class $K_0(k)$ ($0\leqq k<1$) *if and only if*

$$\sum_{n=2}^{\infty}n(n-k)|a_n|\leqq|a_1|(1-k),$$

where $|a_1|\neq 0$.

LEMMA 2. *A function*

$$f(z)=|a_1|z-\sum_{n=2}^{\infty}|a_n|z^n$$

is in the class $S^*_0(k)$ ($0\leqq k<1$) *if and only if*

$$\sum_{n=2}^{\infty}(n-k)|a_n|\leqq|a_1|(1-k),$$

where $|a_1|\neq 0$.

LEMMA 3. *A function*

$$f(z)=|a_1|z-\sum_{n=2}^{\infty}|a_n|z^n$$

is in the class $K_0(k)$ ($0\leqq k<1$), *then for* $z\in U$,

$$|f(z)|\geqq\frac{|a_1|\ |z|\ \{2(2-k)-(1-k)|z|\}}{2(2-k)}$$

and

$$|f(z)|\leqq\frac{|a_1|\ |z|\ \{2(2-k)+(1-k)|z|\}}{2(2-k)},$$

where $|a_1|\neq 0$. *The equality holds for the function*

$$f(z)=|a_1|z-\frac{(1-k)|a_1|}{2(2-k)}z^2.$$

THEOREM 1. *If a function*

$$f(z)=|a_1|z-\sum_{n=2}^{\infty}|a_n|z^n \ (|a_1|\neq 0)$$

is in the class $S^*_0(0)$, *then for* $\alpha>0$ *and* $z\in U$,

$$\frac{|a_1||z|^{1+\alpha}(2-|z|)}{2\Gamma(2+\alpha)}\leqq|D_z^{-\alpha}f(z)|\leqq\frac{|a_1||z|^{1+\alpha}(2+|z|)}{2\Gamma(2+\alpha)}.$$

The equality holds for the function

$$f(z)=|a_1|z-\frac{(2+\alpha)|a_1|}{4}z^2.$$

PROOF. Let consider the function

$$\begin{aligned}F(z)&=\Gamma(2+\alpha)z^{-\alpha}D_z^{-\alpha}f(z)\\&=|a_1|z-\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2+\alpha)|a_1|}{\Gamma(n+1+\alpha)}z^n.\end{aligned}$$

Then, we have the following inequality with the aid of Lemma 2,

$$\begin{aligned}|F(z)|&\leqq|a_1||z|+|z|^2\sum_{n=2}^{\infty}|a_n|\\&\leqq|a_1||z|+|z|^2\frac{|a_1|}{2}\\&=\frac{|a_1|\,|z|}{2}(2+|z|).\end{aligned}$$

In the same way, we have

$$|F(z)|\geqq\frac{|a_1|\,|z|}{2}(2-|z|).$$

Hence, we have the theorem.

COROLLARY 1. *Under the conditions of Theorem* 1, $D_z^{-\alpha}f(z)$ *is included in the disk with center at the origin and radius* $3|a_1|/2\Gamma(2+\alpha)$, *where* $\alpha>0$.

THEOREM 2. *If a function*

$$f(z)=|a_1|z-\sum_{n=2}^{\infty}|a_n|z^n \ (|a_1|\neq 0)$$

is in the class $K_0(k)$, *then for* $\alpha>0$ *and* $z\in U$,

$$\frac{|a_1|\,|z|^{1+\alpha}(4-|z|)}{4\Gamma(2+\alpha)}\leqq|D_z^{-\alpha}f(z)|\leqq\frac{|a_1|\,|z|^{1+\alpha}(4+|z|)}{4\Gamma(2+\alpha)}.$$

The equality holds for the function

$$f(z)=|a_1|z-\frac{(2+\alpha)|a_1|}{8}z^2.$$

PROOF. *Let*

$$F(z)=\Gamma(2+\alpha)z^{-\alpha}D_z^{-\alpha}f(z)$$
$$=|a_1|z-\sum_{n=2}^{\infty}|A_n|z^n.$$

Then, by means of Lemma 1,

$$\sum_{n=2}^{\infty}n|A_n|\leqq\sum_{n=2}^{\infty}n^2|a_n|$$
$$\leqq|a_1|.$$

Hence, the function $F(z)$ *is in the class* $K_0(0)$ *by Lemma* 1. *Therefore, using Lemma* 3, *we have the theorem.*

COROLLARY 2. *Under the conditions of Theorem* 2, $D_z^{-\alpha}f(z)$ *is included in the disk with center at the origin and radius* $5|a_1|/4\Gamma(2+\alpha)$, *where* $\alpha>0$.

THEOREM 3. *If a function*

$$f(z)=|a_1|z-\sum_{n=2}^{\infty}|a_n|z^n \quad (|a_1|\neq 0)$$

is in the class $S^*_0(0)$, *then for* $\alpha>0$ *and* $z\in U$,

$$|D_z^{1-\alpha}f(z)|\leqq\frac{|a_1||z|^{\alpha}}{\Gamma(2+\alpha)}\left(1+\alpha+\frac{2+\alpha}{2}|z|\right).$$

PROOF. Let consider the function

$$F(z)=\Gamma(2+\alpha)z^{-\alpha}D_z^{-\alpha}f(z).$$

Then, using Lemma 2, we have

$$|F'(z)|\leqq|a_1|+|z|\sum_{n=2}^{\infty}n|a_n|$$
$$\leqq|a_1|(1+|z|).$$

Therefore, we have the theorem with the aid of Theorem 1.

COROLLARY 3. *Under the conditions of Theorem* 3, $D_z^{1-\alpha}f(z)$ *is included in the disk with center at the origin and radius* $(4+3\alpha)|a_1|/2\Gamma(2+\alpha)$, *where* $\alpha>0$.

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