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An Example on Compact linear Operators

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Let l^2 be the set $\{(\xi_1, \xi_2, \cdots) \mid \xi_i \in \mathbb{C}$ for $i=1, 2, \cdots$ and $\sum_{i=1}^{\infty} |\xi_i|^2 < \infty\}$, where $\mathbb{C}$ is the field of complex numbers. Then, as is well known l^2 is a complex Hilbert space. We define

$$T : \ell^2 \longrightarrow \ell^2 \qquad (※)$$

by $T((\xi_1, \xi_2, \cdots)) = (\xi_1, \frac{\xi_2}{2}, \cdots)$. In this note, we shall prove some properties with respect to T (propositions 2, -3 and Theorem 5).

Let X and Y be normed spaces. An operator $S : X \to Y$ is called a compact linear operator if it satisfies the following conditions;

(i) S is a linear operator.

(ii) for every bounded subset M of X, $S(M)$ is relatively compact, *i. e.*, $\overline{S(M)}$ is compact in Y.

Lemma 1. Let X and Y be normed spaces. For a linear operator $S : X \to Y$, if S is bounded and $\dim S(X) < \infty$, then S is compact.

Proof. Take a bounded sequence $\{x_n\}$ in X. Since $\|T\| = 1$ and

$$\|Tx_n\| \leq \|T\| \|x_n\|,$$

$\{Tx_n\}$ is bounded. Therefore $\{\overline{Tx_n}\}$ is bounded (and closed). By our assumption $\dim S(X) < \infty$, and thus $\{\overline{Tx_n}\}$ is compact. It follows that $\{Tx_n\}$ has a convergent subsequence. Since

S is compact $\leftrightarrow$ for every bounded sequence $\{x_n\}$ in X, $\{Tx_n\}$ has a convergent subsequence ([1]),

S is compact.

Proposition 2. The linear operator T defined in (※) is compact.

Proof. Define

$$T_n : \ell^2 \longrightarrow \ell^2$$

by $T_n(x) = (\xi_1, \frac{\xi_2}{2}, \cdots, \frac{\xi_n}{n}, 0, \cdots)$ for $x = (\xi_1, \xi_2, \cdots, \xi_n, \cdots) \in \ell^2$. Then T_n

is bounded and linear. Moreover dim $(T_n(\ell^2))=n$. Hence by Lemma 1, T_n is compact. Further-more, for each $x=(\xi_1, \xi_2, \cdots)\in\ell^2$,

$$\|(T-T_n)x\|^2=\sum_{j=n+1}^{\infty}\left|\frac{\xi_j}{j}\right|^2=\sum_{j=n+1}^{\infty}\frac{1}{j}|\xi_j|^2\leq\frac{1}{(n+1)^2}\sum_{j=n+1}^{\infty}|\xi_j|^2$$
$$\leq\frac{\|x\|^2}{(n+1)^2}$$

Taking the supremum over all x with $\|x\|=1$, we see that

$$\|T-T_n\|\leq\frac{1}{n+1}.$$

Therefore $\|T-T_n\|\to 0$ (*i.e.*, $T_n\to T$), and thus T is compact ([2]). If we put

$$e_i=(0\cdots, 0\ \overset{i}{1},\ 0\cdots)$$

then $\{e_1, e_2, \cdots\}$ is an orthonormal base of ℓ^2. Therefore every $x\in\ell^2$ has a unique representation

$$x=\sum_{j=1}^{\infty}\xi_j e_j,$$

and

$$Tx=\sum_{j=1}^{\infty}\frac{1}{j}\xi_j e_j$$

We define an operator

$$\begin{array}{rcl} P_j : \ell^2 & \longrightarrow & \ell^2 \\ \Psi & & \Psi \\ x & \longmapsto & \xi_j e_j \end{array}$$

Proposition 3. Under the above situation we have following.

(i) $\{1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}, \cdots\}$ is the set of eigenvalues of T.

(ii) $\{e_1, e_2, \cdots, e_n, \cdots\}$ is an orthonormal set of eigenvectors of T.

(iii) T is self-adjoint and positive.

(iv) The following holds.

$$T=\sum_{j=1}^{\infty}\frac{1}{j}P_j.$$

Proof. For each $x=(\xi_1, \xi_2, \cdots\cdots, \xi_n, \cdots\cdots)\in\ell^2$ if we put

$$Tx=(\xi_1, \frac{\xi_2}{2}, \cdots, \frac{\xi_n}{n}, \cdots)=\lambda x=(\lambda\xi_1, \lambda\xi_2, \cdots, \lambda\xi_n, \cdots),$$

we have

$$\xi_1=\lambda\xi_1, \cdots\cdots\cdots, \frac{\xi_n}{n}=\lambda\xi_n, \cdots\cdots\cdots.$$

Therefore, if $\lambda=\frac{1}{n}$, then $\xi_i=0$ for $i\neq n$. Since $n=1, 2, \cdots$, we get $\{1, \frac{1}{2}, \frac{1}{3}, \cdots\}$ as the set of eigenvalues of T. It is easy to see that

$$Te_j=\frac{1}{j}e_j,$$

and thus $e_1=(1, 0, \cdots)$, $e_2=(0, 1, 0, \cdots)\cdots$, $e_n=(0, \cdots 0, \overset{n}{1}, 0\cdots), \cdots$ are eigenvectors of T. Moreover, since $e_i \perp e_j$ for $i\neq j$, we have $\{e_1, e_2, \cdots e_n, \cdots\}$ $,\cdots\}$ as an orthonormal set of eigenvector of T.

Therefore (i) and (ii) are proved.

For each $x=(\xi_1, \xi_2, \cdots \xi_n, \cdots)\in \ell^2$

$$(Tx, x)=\sum_{j=1}^{\infty}\frac{1}{j}\xi_j\bar{\xi}_j\geq 0,$$

and $(Tx, x)=0 \leftrightarrow x=0$, where $(\quad,\quad)$ is the inner product defined on ℓ^2. Hence T is positive. For $y=(\eta_1, \eta_2, \cdots\eta_n, \cdots)\in \ell^2$,

$$(Tx, y)=(x, Ty)=\sum_{j=1}^{\infty}\frac{1}{j}\xi_j\bar{\eta}_j.$$

So, it follows that T is self-adjoint.

For (iv) let $x=(\xi_1, \xi_2, \cdots \xi_n, \cdots)\in \ell^2$. Then

$$\|(T-\sum_{j=1}^{m}\frac{1}{j}P_j)x\|^2=\|\sum_{j=m+1}^{\infty}\frac{1}{j}\xi_je_j\|^2=\sum_{j=m+1}^{\infty}\frac{1}{j^2}|\xi_j|^2\leq$$
$$\frac{1}{(m+1)^2}\sum_{j=m+1}^{\infty}|\xi_j|^2\leq\frac{\|x\|^2}{(m+1)^2},$$

so that

$$\|T-\sum_{j=1}^{m}\frac{1}{j}P_j\|\leq\frac{1}{m+1}\rightarrow 0$$

as $m\rightarrow\infty$.

For any real λ we define

$$E_\lambda=\sum_{\lambda_j\leq\lambda}P_j \quad (\lambda\in\boldsymbol{R}), \qquad (※※)$$

which is an one-parameter family of projections, λ being the parameter.

Definition 4. A real ***spectral family*** is a one-parameter family $\xi=(E_\lambda)_{\lambda\in R}$ of projections E_λ defined on a Hilbert space H which depends on a real parameter λ and is such that

(a) $E_\lambda\leq E_\mu$ hence $E_\lambda E_\mu=E_\mu E_\lambda=E_\lambda$ for $\lambda<\mu$.

(b) $\lim_{\lambda\rightarrow-\infty}E_\lambda x=0$ for $x\in H$.

(c) $\lim_{\lambda\rightarrow+\infty}E_\lambda x=x$ for $x\in H$.

(d) $E_{\lambda+0}\ x=\lim_{\mu\to\lambda+0} E_{\mu}x=E_{\lambda}x,$

where $\mu\to\lambda^{+0}$ means that we let μ approach λ from the right.

Then $\{E_{\lambda}\}_{\lambda\in R}$ defined as (※※) is a spectral family associated with the bounded self-adjoint linear operator T.

Theorem 5. T has the spectral representation

$$T=\int_{0-0}^{1}\lambda d E_{\lambda},$$

where $\mathcal{E}=(E_{\lambda})_{\lambda\in R}$ is the spectral family associated with T.

Proof. At first we have note that

$$0=\inf_{\|x\|=1}(Tx,\ x),\quad 1=\sup_{\|x\|=1}(Tx,\ x).$$

We choose a sequence $\{\mathcal{S}_n\}$ of partions of $(a,\ b]$, where $a<0$ and $1<b$. That is, every $\mathcal{S}_n$ is a partion of $(a,\ b]$ into intervals

$$\triangle_{n,j}=(\lambda_{n,j},\ \mu_{n,j}),\quad j=1,\ 2,\cdots n.$$

of length $\ell(\triangle_{n,j})=\mu_{n,j}-\lambda_{n,j}$ Here $\mu_{n,j}=\lambda_{n,j+1}$ for $j=1,\cdots,n-1$. In particular, the sequence $\{\mathcal{S}_n\}$ is such that

$$\eta(\mathcal{S}_n)=\max_j\ \ell\ (\triangle\mu_{n,j})\to 0 \qquad (***)$$

as $n\to\infty$. We put

$$E(\triangle_{n,j})=E_{\mu,j}-E_{n,j},$$

then we have

$$\lambda_{n,j}\ E(\triangle_{n,j})\leq TE\ (\triangle_{n,j})\leq\mu_{n,j}\ E(\triangle_{n,j})$$

([1]). By summation over j from 1 to n, for every n we get

$$\sum_{j=1}^{n}\lambda_{n,j}E(\triangle_{n,j})\leq\sum_{j=1}^{n}TE(\triangle_{n,j})\leq\sum_{j=1}^{n}\mu_{n,j}E(\triangle_{n,j}) \qquad (****)$$

Since

(i) $\mu_{n,j}=\lambda_{n,j+1}$ for $j=1,\cdots n-1$.

(ii) $\lambda<0\Longrightarrow E_{\lambda}=0$

(iii) $n\geqq 1\Longrightarrow E_{\lambda}=I$ (identity operator),

We simply have

$$T\sum_{j=1}^{n}E(\triangle_{n,j})=T\sum_{j=1}^{n}(E\mu_{n,j}-E\lambda_{n,j})=T(I-0)=T.$$

Formula (***) implies that for every $\mathcal{E}>0$ there is an n such that $\eta(\mathcal{S}_n)<\varepsilon$, and thus in (****) we have

$$\sum_{j=1}^{n}\mu_{n,j}E(\triangle_{n,j})-\sum_{j=1}^{n}\lambda_{n,j}E(\triangle_{n,j})=\sum_{j=1}^{n}(\mu_{n,j}-\lambda_{n,j})E(\triangle_{n,j})\leqq\varepsilon I.$$

From this and (＊＊＊＊), given any $\xi > 0$ there is an N such that for every $n > N$ and every choice of $\hat{\lambda}_{n,j} \in \triangle_{n,j}$ we have

$$\| T - \sum_{j=1}^{n} \hat{\lambda}_{n,j} E(\triangle_{n,j}) \| < \varepsilon .$$

References

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