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SOME PROPERTIES OF CONNECTED-OPEN TOPOLOGY FOR FUNCTION SPACES

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§ 1. INTRODUCTION

S. A. Naimpally (3) proved that the compact-open topology k for function spaces is contained in the graph topology Γ and also *A.* Irudayanathan and *S. A.* Naimpally (4) showed that if X is locally connected, then the campact-open topology k for the continuous function spaces is smaller than connected-open topology T.

In this paper, We show that if X is locally connected then the graph topology is smaller than the connected-open topology (Theorem 3.6). And we compare T with some of the other well-known topologies for function spaces.

Therefore, all the above can be summarized by the following relation : the pointwise convergence $(p.\ c)$ topology $\subset$ the k-topology $\subset$ the Γ-topology $\subset T$-topology.

§ 2. DEFINITION AND PRELIMINARIES

DEFINITION 2. 1. Let X and Y be topological spaces and let F denote the set of all functions on X to Y. Let C denote the subset of F consisting of all continuous functions.

For $f \in F$, the graph of f, denoted by $G(f)$, is the set

$$\{ (x, f(x)) \mid x \in X \} \subset X \times Y.$$

Let $X \times Y$ be assigned the usual product topology.

A function $f \in F$ is called almost continuous iff for each open set U in $X \times Y$ containing $G(f)$, there exists a $g \in C$ such that $G(g) \subset U$.

Corresponding to each open set U in $X \times Y$, let

$$F_U = \{ f \in F \mid G(f) \subset U \}.$$

The topology induced on F by a basis consisting of sets of the form $\{ F_U \}$ for each open set U in $X \times Y$ is called the graph topology Γ for F.

EXAMPLE 2.2. Let A be the subset of F consisting of all almost continuous functions. Whereas every continuous function is almost continuous, there exist almost continuous functions which are not continuous. If $X=Y=$the set of all real numbers with the usual topology, then the function $f\in F$ defined by

$$f(x)=\begin{cases} \sin\frac{1}{x} & \text{for } x\neq 0 \\ 0 & \text{for } x=0 \end{cases}$$

is almost continuous but not continuous.

THEOREM 2.3. If Y contains at least two points, then the following are equivalent : (i) X and Y are T_1-spaces, (ii) (F, Γ) is T_1.

PROOF. First let X and Y be T_1-spaces and let $f, g\in F$ and $f\neq g$. Then there exists an $a\in X$ such that $f(a)\neq g(a)$. Since Y is T_1 there exists an open set U in Y such that $f(a)\in U$ but $g(a)\notin U$. Also since X is T_1, $(X-\{a\})\times Y$ is open in $X\times Y$. Consequently, $(X\times U)\cup(X-\{a\})\times Y$ contains $G(f)$ but not $G(g)$. So (F, Γ) is T_1.

Next let (F, Γ) be T_1. If Y is not T_1, there exist distinct points $p, q\in Y$ such that every open set which contains p also contains q. Let $f, g\in F$ be such that $f(x)=p$, $g(x)=q$ for all $x\in X$. Then every open set in $X\times Y$ which contains $G(f)$ also contains $G(g)$ and so F is not T_1, a contradiction. Thus Y is T_1. To show that X is T_1, assume this is false. Then there exist $a, b\in X$ such that each open set in X containing a also contains b. Let p, q be two distinct points of Y, Define $f, g\in F$ such that $f(b)=q$, $g(b)=p$, $f(x)=g(x)=p$ for all $x\in X$, $x\neq b$. Then every open set in $X\times Y$ which contains $G(f)$ also contains $G(g)$ and so (F, Γ) is not T_1, a contradiction.

This shows the equivalence of (i) and (ii).

DEFINITION 2.4 : For each connected subset K of X and each pair of open subsets U, V of Y, let

$$W(K; U, V)=\{f\in F \mid f(K)\subset U\cup V,\ f(K)\cap U\neq\phi\neq f(K)\cap V\}.$$

The topology T generated by the subbasis $\{W(K; U, V)\}$ is called the connected-open topology for F.

And a function $f: X\to Y$ is a connected function iff for each connected set $K\subset X$, $f(K)$ is a connected subset of Y.

THEOREM 2. 5. If Y is completely normal, then Co consisting of all connected functions is closed in (F, T).

PROOF) Suppose $f \in F$ is a limit point of Co but $f \in Co$. Then there exists a connected set $K \subset X$ such that $f(K)$ is not a connected subset of Y. Since Y is completely normal, there exist nonempty disjoint open subset U, V of Y such that $f(K) \subset U \cup V$, $f(K) \cap U \neq \phi \neq f(K) \cap V$. Since f is a limit point of Co, there exist a $g \in Co$ such that $g \in W(K; U, V)$ which is an open neighborhood of f. So $g(K) \subset U \cup V$, $g(K) \cap U \neq \phi \neq g(K) \cap V$ but $U \cap V = \phi$ which contradicts the fact that $g(K)$ is connected. so $f \in Co$ and Co is closed in (F, T).

§ 3. COMPARISON OF CONNECTED OPEN TOPOLOGY WITH THE OTHER TOPOLOGIES.

THEOREM 3. 1. The $p.c$ topology $\subset$ the k-topology $\subset$ the Γ-topology when X is T_2.

PROOF. A subbase for the compact-open topology k on F consists of the family of sets of the form

$$W(M. U) = \{ f \in F \mid f(M) \subset U,\ M \text{ compact in } X,\ U \text{ open in } Y \}.$$

Since every singleton is a compact set it is clear that $p.c. \subset k$. If $W(M. U)$ is a subbase element for k, then the set

$$V = (X \times U) \cup ((X - M) \times Y)$$

is open in $X \times Y$, and $F_V = W(M, U)$.

THEOREM 3. 2. Let X and Y be uniform spaces with uniformities u and v respectively. If X is compact then the uniform convergence $(u.\ c.)$ topology is equivalent to the graph topology for C.

PROOF. Each $f \in C$ is uniformly continuous since X is compact and so in view of Theorem 3. 2, it is sufficient to prove that the graph topology is contained in the $u.\ c.$ topology for C. Let $f \in C$ and let U be an open set in $X \times Y$ containing $G(f)$. Since $f \in C$, $G(f)$ is homeomorphic to X and so $G(f)$ is compact. Hence there exists a $V_1 \in u$ and a $V_2 \in v$ such that $\cup_{x \in X} \{V_1[x] \times V_2[f(x)]\} \subset U$. (2). Now if $g \in C$ and $g \in W(V_2)[f]$ then $g(x) \in V_2[f(x)]$ for all $x \in X$ and so $G(g) \subset U$. This shows that C is open in the $u.\ c.$ topology for C.

THEOREM 3. 3. The *p. c.* topology for F is smaller than T for F.

PROOF) A subbase for the *p. c.* topology consits of the set $\{ (x, U) \}$ where

$(x, U) = \{ f \in F \mid f(x) \in U$ where $x \in X$ and U is open in $Y \}$.

It is easy to see that $(x, U) = W(\{x\}; U, U)$ and so is open in T which proves the theorem.

THEOREM 3.4. If X is locally connected then the k-topology for C is smaller than T. (2).

PROOF. A subbasis for the k-topology for C consists of sets of the form

$W(M, U) = \{ f \in C \mid F(M) \subset U$, M compact in X and U open in $Y \}$.

Let $h \in W(M, U)$. Then $M \subset h^{-1}[U]$ which is open in X. Since X is locally connected and M compact,

$$M \subset \cup_{i=1}^{n} V_i \subset h^{-1}[U]$$

where each V_i is a nonempty open connected subset of X. Clearly

$$\cap_{i=1}^{n} W(Vi; U, U)$$

is an open T-neighbourhood of h which is contained in $W(M, U)$. Thus $W(M, U)$ is open in T which proves the theorem.

COROLLARY 3. 5. If X is locally connected then the k-topology is smaller than the graph topology for F. And the proof is similar to that of (Theorem 3. 4.)

THEOREM 3. 6. If X is locally connected then the graph topology is smaller than the connected-open topology.

PROOF) Let X be locally connected. Then for all $x \in X$, there exist at least a connected-open neighbourhood U such that $x \in U \subset X$. If we define

$B = \{ U \times V \mid U$ is a connected-open neighbourhood at x for each $x \in X$ and V is open in $Y \}$.

Then B is a basis for $X \times Y$. Therefore a member of the basis for the graph topology $F_U = \{ f \in F \mid G(f) \subset U \times V \}$ conicides with a member of the subbasis for the connected-open topology

$$W(U; V, V) = \{ f \in F \mid f(U) \subset V \cup V \}.$$

COROLLARY 3. 7. Let X be locally connected. Then we have the following relation: the *p. c.* topology $\subset$ the *k*-topology $\subset$ the Γ-topology $\subset$ the T-topology.

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