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ON LOCALLY B*-EQUIVALENT ALGEBRAS

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ABSTRACT

Let A be a Banach *-algebra and $C(t)$ be a closed *-subalgebra of A gengerated by $t\in A$. A is locally B^*-equivalent [B^*-equivalent] if $C(t)$ [A] for every hermitian element t is *-isomorphic to some B^*-algebra. It was proved that the locally B^*-equivalent algebras with some conditions is B^*-equivalent by B. A. Barnes.

In this paper, we obtain the some conditions for a locally B^*-equivalent algebra to be B^*-equivalent.

§ 1. Introduction

Through this paper, A is a complex Banach*-algebra. A is B^*-equivalent if A is *-isomorphic to some B^*-algebra, and locally B^*-equivalent if for a hermitian element t of A, the closed *-subalgebra $C(t)$ generated by t is B^*-equivalent. If A is B^*-equivalent, then *-isomorphism on A is a homeomorphism and hence A has the algebraic and the topological structure of B^*-algebra. B. A. Barnes proved that the locally B^*-equivalent algebras with some conditions is B^*-equivalent. [1] In this paper we prove the followings;

(1) If A is locally B^*-equivalent and $\|x^*x\|\geqq k\|x\|\|x^*\|$ for some $k>0$, then A is B^*-equivalent.

(2) If A is locally B^*-equivalent and $\rho(x^*x)\leqq\rho(x)^2$ for all $x\in A$, then A is B^*-equivalent.

(3) Every maximal normal subset of a locally B^*-equivalent algebra is B^*-equivalent.

§ 2. Notation and definitions

Let A be a normed *-algebra over the complex field with conjugate linear involution $a\longmapsto a^*$. The spectrum and spectral radius of an element x of A will be denoted by $sp(x)$ and $\rho(x)$ respectively. For hermitian elements h, k (*i.e.* $h^* = h$ and $k^* = k$), we write $h > k$ [$h \geqq k$] when $sp(h-k) \subset (0, \infty)$ [$sp(h-k) \subset [0, \infty)$].

The involution is hermitian if each hermitian element has real spectrum, and symmetric if each element of the form x^*x has nonnegative real spectrum (or for every $x \varepsilon A$, $e + x^*x$ has an inverse in A). An algebra with symmetric involution is called to be symmetric.

A linear functional f on A is said to be positive if $f(x^*x) \geqq 0$. for all $x \varepsilon A$. The *-radical of A is the *-ideal of all elements vanishing under every positive functional with $f(e) = 1$ and is denoted by R_A. That A is *-semisimple means *-radical $R_A = \{0\}$.

If f, is a positive functional on A, then $I_f = \{x \varepsilon A \mid f(x^*x) = 0\}$ is a left ideal of A. Set $P = \cap I_f$, where the intersection is over all positive functionals on A. Then P is two sided ideal [7. Pro. 3. 1]. A is P-commutative if $xy - yx \varepsilon P$ for all x, $y \varepsilon A$.

A *-homomorphism $\psi : A \rightarrow \mathcal{L}(H)$ is called a *-representation of A on Hilbert space H, where $\mathcal{L}(H)$ is the set of all condinuous linear maps on H. f is representable if there exists a *-representation of A on H and a cyclic vector $\eta \varepsilon H$ such that $f(x) = (\psi(x)\eta \mid \eta)$ for all $x \varepsilon A$.

For every Banach *-algebra A, it is true $R_A = \cap I_f$, f representable. Hence $P \subset R_A$. In particular, if A has an identity, then every positive functional on A is representable [6. Th 1. 6] and hence $P = R_A$.

§ 3. Some results on locally B*-equivalent algebras.

The Banach *-algebra A_1 obtained by adjointing anidentity to A is locally B^*-equivalent iff A is locally B^*-equivalent. Hence we may assume that A has an identity.

Proposition 3. 1 Assume that A is locally B^*-equivalent.

Then (1) * is hermitian and hence symmetric

(2) A is *-semisimple

(3) There exists a unique norm $|\cdot|$ on A which has the B^*- property, $|a^*a| = |a|^2$ for all $a \in A$, *i.e.* A is A^*-algebra. In particular, * is a proper involution (*i.e.* $a^*a = 0$ implies $a = 0$).

Proof) Let t be a hermitian element of A. Then the spectrum of t in $C(t)$ is real, since $C(t)$ is *-isomorphic to some B^*-algebra [2.58. 4]. Since $sp_A(t) \subset sp_{C(t)}(t) \subset R$, the spectrum of t in A is real. Therefore * is a hermitian involution. Hence * is symmetric by Shirali's Theorem [5. Th. 1]. This proves (1).

The *-radical R_A is a closed *-ideal of A. For a hermitian element t in R_A, $C(t) \subset R_A$ and $sp(t) = \{0\}$. Since $C(t)$ is B^*-equivalent, then $t=0$, and $R_A = \{0\}$, *i, e.* A is *-semisimple.

Since A is *-semisimple, there exists a norm $|\cdot|$ on A which has the B^*-property, $|a^*a| = |a|^2$ for all $a \in A$ [3. Th. 4. 7. 16]. Suppose that $|\cdot|_1$ is another norm on A with the B^*-property. Given $a \in A$, $C(a^*a)$ is B^*-equivalent. Thus $|a^*a| = |a^*a|_1$. Since $|a|^2 = |a^*a| = |a^*a|_1 = |a|_1^2$, then $|a| = |a|_1$.

Since $|a^*a| = |a|^2$ for all $a \in A$, the involution is a isometry with respect to the auxiliary norm $|\cdot|$. Hence * is continuous with respect to $|\cdot|$. If $x_n \to a$ and $x_n^* \to b$ implies $b = a^*$, then, from the closed graph theorem, * is continuous with respect to $\| \quad \|$. For all $x \in A$.

$$|b - a^*| \leqq |b - x_n^*| + |x_n^* - a^*|$$

and

$$|x|^2 \leqq \rho_A(x^*x) \leqq \|x^*\| \|x\|,$$

by [3. Th. 4 . 1 . 14], so that

$$|b - x_n^*|^2 \leqq \|b - x_n^*\| \quad \|b^* - x_n\|$$

and

$$|x_n^* - a^*|^2 \leqq \|x_n^* - a^*\| \|x_n - a\|$$

Since $\|b - x_n^*\| \to 0$ and $\|x_n^* - a^*\| \to 0$ as $n \to \infty$, then $|b - x_n^*| \to 0$ and $|x_n^* - a^*| \to 0$. $|b - a^*| = 0$ and hence $b = a^*$.

Proposition 3. 2 Assume that A is a commutative Banach*-algebra which is locally B^*-equivalent. Then A is B^*-equivalent.

Proof) Since A is a commutative, *-semisimple Banach*-algebra with hermitian involution by Pro. 3. 1(1), B is B^*-equivalent, if for every continuous real function ϕ on domain $\mathscr{D} \subset R$, $\phi \circ \hat{a} \in \hat{A}$ whenever the range of $\hat{a}$ is in $\mathscr{D}$ where $\hat{A}$ is the set of Gelfand transforms of all $a \in A$, by Katznelson's Theorem. Hence we show that for a continuous real function ϕ with real domain, $\phi \circ \hat{a} \in \hat{A}$. If t

$\in A$ and $\hat{t}$ has range in $\mathscr{D}$, then $t=t^*$ by (1) and (2). Thus $\hat{C}(t)$ is a complete *-subalgebra of $\hat{A}$ with respect to a sup norm. $\phi_0 \hat{t} \in \hat{A}$ by [3. Th. 4. 8. 7]. Hence A is B^*-equivalent.

Proposition 3. 3. Assume that A is locally B^*-equivalent and B Banach *-algebra. If $\phi : A \to B$ is a continuous *-homomorphism, then $\phi(A)$ is locally B^*-equivalent. In particular, if I is a closed *-ideal, then A/I is locally B^*-equivalent.

Proof) Let h be a hermitian element of $\phi(A)$. There exists $a \in A$ such that $\phi(a)=h$, and since ϕ is *-homomorphism, $\phi(a)=h=h^*=\phi(a^*)$, and so $a-a^* \in \mathrm{Ker}\phi=K$. K is a closed *-ideal since ϕ is continuous. Set $t=\frac{a+a^*}{2}$. then $a-t=\frac{a-a^*}{2} \in K$, and so $h=\phi(t)$ for hermitian element t of A. Let $C=\phi(C(t))$. Define $\psi=\phi\mid_{C(t)} : C(t) \to C$. Then Ker $\psi=K \cap C(t)$. Hence $C(t)/K \cap C(t)$ is $*$-isomorphic to C. Since $C(t)/K \cap C(t)$ is B^*-equivalent, C is B^*-equivalent. Since $h=\phi(t) \in C$ and $C(h)$ is a closed *-subalgebra of C, $C(h) \subset C$. Thus $C(h)$ is B^*-equivalent and so $\phi(A)$ is locally B^*-equivalent.

§ 4. Main Theorems

Theorem 4. 1 Assume that A is locally B^*-equivalent with

$$\| x^*x \| \geqq k \| x \| \ \| x^* \| \qquad (x \in A)$$

for some $k>0$. Then A is B^*-equivalent.

Proof) By Pro. 3. 1(1), a hermitian element h has a real spectrum. Since $\| h^*h \| \geqq k \| h \| \| h^* \|$, then $\| h^{2n} \| \geqq k \| h^{2^{n-1}} \|^2$ by Rickart's Theorem, and so $\rho(h) \geqq k \| h \|$. Let N be the set of all $a \in A$ such that $sp(x^*x) \subset (-\infty, 0]$, *i. e.* $-x^*x \geqq 0$. Let $x \in N$. Then $-x^*x \geqq 0$ and $-xx^* \geqq 0$ since $sp(xx^*)=sp(x^*x)$ by [2. *Ex.* 51. 16]. We write $x=h+ik$, h, k hermitian elements of A. Since $2h^2 \geqq 0$ and $2k^2 \geqq 0$ by [2. Le. 53. 3],

$$x^*x+xx^*=2h^2+2k^2 \geqq 0$$

$$x^*x=2h^2+2k^2-xx^* \geqq 0$$

Thus $sp_A(x^*x)=\{0\}$. Since A is locally B^*-equivalent,

$$|x|^2=|x^*x| \leqq \rho_A(x^*x)=0$$

by [3. Th. 4. 1. 14]. Hence $x=0$, *i. e.* $N=\{0\}$. By [8. Le. 2. 6] A is a B^*-algebra in an equivalent norm, and hence A is *-equivalent.

Theorem 4. 2 Assume that A is locally B^*-equivalent and $\rho(x^*x) \leqq \rho(x)^2$ for all $x \in A$. Then A is B^*-equivalent.

Proof) Since every positive functional on A with an identity is representable, the *-radical $R_A = \cap \mathrm{Ker} f$, where f is positive functional on A. Then for all $x \in A$,

$$|f(x)| \leqq f(e) \rho(x^*x)^{\frac{1}{2}} \leqq f(e) \rho(x)$$

by [4. Th. 1. 31] and the hypothesis. For all x, $y \in A$, $f(xy) = f(yx)$. Hence $xy - yx \in \cap \mathrm{Ker} f = R_A = P$ for all x, $y \in A$. Thus A is P-commutative. Since A is *-semisimple by Pro. 3. 1(2), A is P-commutative iff A is commutative. [7] Thus by Pro. 3. 3, A is B^*-equivalent.

Theorem 4. 3 Every maximal normal subset of a locally B^*-equivalent algebra is B^*-equivalent.

Proof) Let N be a maximal normal subset of A. Then $N \cup N^*$ is nomal subset of A. Since N is maximal, $N^* = N$. That N is *a* *-subalgebra is clear, and hence maximal commutative *-subalgebra. Let $\{x_n\}$ be a sequence of N which converges to an element $x \in A$. All $x_n's$ commute with every element of N. Thus x commutes with every element of N. Since N is *-subalgebra, x^* commutes with every element of N. For all n, $x_n x^* = x^* x_n$ and

$$xx^* = \lim_{n \to \infty} x_n x^* = \lim_{n \to \infty} x^* x_n = x^* x.$$

$x \in N$ and hence N is closed, *i. e.* N is a closed, maximal, commutative *-algebra. N is B^*-equivalent by [1]

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