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Properties of finite spaces

By Yong-Guk Kim

In this paper several mathematical structures are studied such as topologies, partial ordering, logical implications and matrices etc, under the unified viewpoint of category on condition that the objects are finite sets, a fairly amount of generalizations are possible.

1. P-ordering and p-q metric

1 − 1 **Definition** : (X, P) is a preordered space X with preorder P iff the following relations are satisfied

(1) $\forall x \in X : xPx$

(2) $\forall x, y, z \in X : xPy$ and yPz implies xPz

furthermore, P is partially ordered iff

(3) $\forall x, y \in X : xPy$ and yPx implies $x = y$

1 − 2 **Definition** : (X, d) is a p-q metric-space iff

(1) $\exists d : d(x, y) \geqq 0$

(2) $\forall x, y, z \in X \exists d : d(x, y) \leqq d(x, z) + d(z, y)$

In particular, the range of the 'characteristic' p-q metric consists of 0 and 1.

From the above diffinition, we can see, in some cases, there is a relation between the characteristic p-q metric and the partial ordering.

1 − 3 **Remark** : Let (X,P) be a preordered space, then there exists a characteristic p-q metric dp; conversely, if a characteristic p-q metric d is given, then there exists a preordering P_d, where xP_dy iff $dp(x, y) = 1$, otherwise 0. and

$d(x, y) = 1$ iff xP_dy

$= 0$ iff xP_dy (*i. e.* x is not related to y with respect to the relation P_d)

Consequently, the following is obtained : $P_{d_P} = P$ and $dp_d = d$ if there is given any finite set $S = \{x, \cdots, x_n\}$ and let $B(x_i) = \{x_j : d(x_i, x_j) = 0\}$ for a characteristic p-q metric d, $B(x_i)$ is the minimal open base containing x_i Adjoining ϕ to the family of all distinct minimal open sets $\{B(x_i) : i \in (1, 2, \cdots, n)\}$, we obtain a base for a topology $\mathfrak{T}_d$.

1 - 4 **Definition** : Let (X, P_1) and (Y, P_2) be preordered sets, then

$$\Phi : (X, P_1) \to (Y, P_2)$$

is called isomorphism iff Φ is bijective and

$$x P_1 y \leftrightarrow \Phi_{(x)} P_2 \Phi_{(y)}$$

Two preordered sets are called isomorphic iff there exists an isomorphism between them; an isomorphism of any preordered set with itself is called an automorphism.

Notation : P^c is the converse relation of P, which is defined as

$$x P^c y \quad \text{iff} \quad y P x$$

Then, we have the following relation.

1 - 5 **Theorem** : The converse of any partial ordering is itself a partial ordering.

For each characteristic p-q metric d, there exists the conjugate characteristic p-q metric d' which is defined as $d(x, y) = d'(y, x)$. Thus, we obtain $\mathfrak{J}_{d'}$, the conjugate topology of $\mathfrak{J}_d$, and therefore,

1 - 6 **Corollary** : $\mathfrak{J}_{d'}$ is equivalent to the topology generated by dp^c (provided that $\mathfrak{J}_d$ is the conjugate topology of $\mathfrak{J}_d$.)

2. Finite topological spaces

Let $S = \{x_1, \cdots, x_n\}$, then by the Kuratowaski's closure axiom, the following is easily obtained.

2 - 1 **Lemma** : Let $(S, \mathfrak{J}_d)$ be a topological space, equipped with the topology $\mathfrak{J}_d$, then $\forall A \subset S : \bar{A} = \bigcup_{x \in A} \{\bar{x}\}$, $x_i \in S$

2 - **Definition** : Define a characteristic function d_{ij},

$$d_{ij} = d(x_i, x_j) = 1, \text{ if } x_j \in \{\overline{x_i}\}$$
$$= 0, \text{ otherwise}$$

Then,

2 - 3 **Remark** :

(1) $\forall x_i \in S : x_i \in \{\overline{x_i}\}$ (reflexitivity)

In particular,

$\forall i \in I$ (index set) : $d_{ij} = 1$

(2) $x_j \in \{\overline{x_i}\}$ and $x_k \in \{\overline{x_j}\}$ implies $x_k \in \{\overline{x_i}\}$

which implies,

if $d_{ij}=1$ and $d_{jk}=1$, then $d_{ik}=1$ (transitivity)

2 - 4 **Theorem** : Let (X, β) be a topological space generated by $\{B_i\}$, then β is T_o-space iff $\beta(x_i)=\beta(x_j)$ implies $x_i=x_j$, where $\beta(x_i)$ is a ***nbbd*** of x_i.

Proof : Assume $x_i \neq x_j$, then the T_o-ness implies either $x_i \notin \beta(x_j)$ or $x_j \notin \beta(x_i)$; consequently $\beta(x_i) \neq \beta(x_j)$.

Conversely, if $\beta(x_i)=\beta(x_j)$ implies $x_i=x_j$ and assume $x_i \notin \beta(x_j)$ and $x_j \notin \beta(x_i)$.

then, $\beta(x_i) \subset \beta(x_j) \subset \beta(x_i)$

Which implies $\beta(x_i)=\beta(x_j)$, and recalling the hypothesis, we obtain $x_i=x_j$.

As a result of the above thorem, we have the following

2 - 5 **Theorem** : (1) If (X, β) is generated by a characteristic function d, and it is T_o, then relation P_d is a partial ordering, (2) If (X, P) is partially ordered, then $(X, \mathcal{J}_{dp})$ is a T_o-space, where $\mathcal{J}_{dp}$ is the topology induced by d_p.

Proof : (1) $d(x, y)=0$ implies xPy. If $d(x, y)=0$ and $d(y, x)=0$, then $\beta(x)=\beta(y)$.

By the T_o-ness $x=y$, therefore yPx and xPy implies $x=y$.

(2) xPy and yPx implies $\beta dp(x)=\beta dp(y)$, and $x=y$

2 - 6 **Definition** : A relation P is defined as

$$x_i P x_j \quad \text{iff} \quad d_{ij}=1$$

2 - 7 **Theorem** : For each characteristic p-q space there exists a preordered set.

3. Matrix

Let $S=\{x_1, x_2, \cdots, x_n\}$, and $D=\{d_{ij}\}$ be the matrix corresponding to a topology $\mathcal{J}$, and let F_i, $B_j \in X$ such that

$$F_i=\{x_i\} : \widetilde{F}_i=\{(x_1, d_{i1}), (x_2, d_{i2}), \cdots, (x_n, d_{in})\}$$

$$B_j=\{x_i\} : B_j=\{(x_1, d_{1j}), (x_2, d_{2j}), \cdots, (x_n, d_{nj})\}$$

then,

3 - 1 **Remark** :

$$x_i \in F_i \quad \text{iff} \quad d_{ii}=1 \text{ iff } x_i \in \{\overline{x_i}\}$$

and

$$x_j \in F_i \text{ iff } d_{ij} = \text{iff } \forall_i : x_i \in B_j$$

3 − 2 **Theorem** : For each j, B_j is the minimal open set containing x_i.

Proof. Let $x_i \in B_j^c$ and $x_k \in F_i$, then $d_{ij} = 0$ and $d_{ik} = 1$, and hence $d_{kj} = 0$. Therefore

$$F_i \subset B_j^c$$

Conversely,

Let u be any open set containing x_j. If $x_k \in U^c$, then $F_k \subset U^c$ and $x_j \notin F_k$. Hence $x_k \notin B_j$. Therefore $U^c \subset B_j^c$

Here, we are particularly concerned about a matrix which consists of 0 and 1, and let this type of matrices be called the characteristic matrix.

3 − 3 **Definition** : Let $M_c = (d_{ij})$, then the transpose matrix of it is defined as

$$M_c^T = (d_{ji})$$

3 − 4 **Remark** : The rows of $M_c^T = (d_{ij})$ are the columns of M_c and the rows of M_c are the columns of M_c^T. In particular, if $M_c = M_c^T$ these matrices are called symmetric and are characterized by the condition that

$$d_{ij} = d_{ji}$$

3 − 5 **Theorem** : A symmetric M_c corresponds to a topology $\mathcal{T}$ iff $M_c^2 = M_c$

3 − 6 **Theorem** : For each p-q metric space, there exists a unique correspondence with M_c.

3 − 7 **Theorem** : Let M_c be the characteristic matrix corresponding to a finite topology $\mathcal{T}_c$, then M_c is isomorphic to a triangular matrix iff $\mathcal{T}_c$ is T_0.

Proof. Assume M_c is isomorphic to a triangular matrix, then $d_{ij} \cdot d_{ji} = 0$ for all $i \neq j$. Let $\mathcal{T}_c$ be a T_o space. Then, there exists a permutation matrix π such that

$$M^* = \pi M_c \pi^T$$

with the condition of nonincreaseness of column sum.

Assume M^* is not triangular, then there exists $i<j$, $d_{ij}^* = 1$ (*i. e.* $M^* = (d_{ij}^*)$)

Thus, $B_i^* \subset B_j^*$ (*i. e.* B_i^* is i th row vectors of M_c^*), and hence $B_i^* \neq$

B_j, which is a contradiction.

4. Logics and graphs

Let a comparison on preordered relation be defined as follows.

4 − 1 **Definition** : Let $P_1 \leq P_2$, which corresponds to the logical concept "P_1 implies P_2", mean that

$$x_i P_1 x_j \text{ implies } x_i P_2 x_j$$

Thus,

4 − 2 **Remark** : In a finite case

$$a_{ij} \leqq a'_{ij} \quad \text{for all } i,\ j$$

since

$$a_{ij} = 1 \quad a_i P_1 a_j \Rightarrow a_i P_2 a_j \Rightarrow a'_{ij} = 1$$

Then, we obtain a fundamental operation of multiplication of relation, defined as follows.

4 − 3 **Definition** : $x_i P_1 P_2 x_k$ means $x_i P_1 x_j$ and $x_j P_2 x_k$ for some $j \in I$ (index set)

4 − 4 **Lemma** : A relation P is transitive iff $P^2 \leqq P$.

By the duality of B_j and F_i in chap. 3, we can obtain the inverse relation P^{-1} for each P in a finite topological space, such that PP^{-1} $P^{-1}P = e$,
where 'e' is the equality relation.

4 − 5 **Definition** : "a precedes b" iff aPb and no x exists such that aPx and xPb.

By this concept we lead to a graphical expression of any partial ordering on S, and also it is easily seen that any partially ordered set is defined up to isomorphism by its diagram, *i, e*

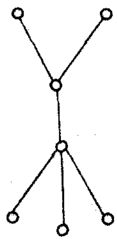
aPb iff there exist $x_0, x_1, \cdots, x_n$ such that $a = x_0$, $b = x_n$ and x_{i-1} precedes to x_i.

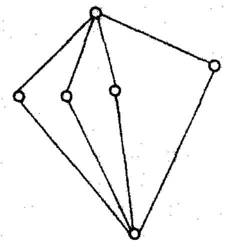
Consequently we have a correspondence between some graphs and partially ordered sets in the abstract spaces.

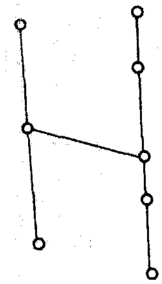
Let categories $C_1 \sim C_6$ be defined respectively as follows,

C_1 : finite sets and set functions

C_2 : finite topological spaces and continuous functions







C_3 : characteristic matrix spaces and martrix homomorphism

C_4 : preorded set on X and order homorphism

C_5 : logical spaces and logical functions

Then, we can define a suitable covarient functors with the leftadjoint.

In particular, inverse partial ordering and dual topologies which are generated by dual characteristic ρ-q metric are well described in terms of categories.

This indicates a usefulness of the categorical method in logics and combinatorics; particularly so, when, partial ordering theory is equipped with graph theory.

Furthermore, for small n a good advantage is exprected for counting problems.

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Mokpo college