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REMARKS ON CR-SUBMANIFOLDS OF A COMPLEX PROJECTIVE SPACE

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§ 1. Introduction

Recently, several authors have defined and studied the CR-submanifolds of a Kaehlerian manifold and the contact CR-submanifolds of a Sasakian manifold (See [1], [2], [5], [6] and [7]). In particular, Yano and Kon [6] studied contact CR-submanifolds of a $(2m+1)$-dimensional unit sphere S^{2m+1}. Furthermore, Pak [5] studied the following theorem by using the theory of Riemannian fibre bundles.

Theorem A. *Let M be an n-dimensional complete CR-submanifold of* CP^m *with semi-flat normal connection and parallel f-structure* in the normal bundle. *If the mean curvature vector of M is parallel and if* $A^e_{bx}f^a_e = f^e_b A^a_{ex}$ *is valid at any point of M, then M is* $CP^{n/2}$ *or M is*

$$\tilde{\pi}(S^{m_1}(r_1)\times\cdots\times S^{m_k}(r_k)),\ n+1=\sum_1^k m_i,\ \ 2\leqq k\leqq n+1,\ \ \sum_1^k r_i^2=1.$$

The purpose of this paper is to project the quanties of $\tilde{\pi}^{-1}(M)$, which is concerned with locally symmetry, onto M and to determine CR-submanifolds of a complex projective space CP^m with the corresponding conditions, where $\tilde{\pi}$ is the submersion defined by the Hopf-fibration: $S^{2m+1}\rightarrow CP^m$

§ 2. CR-submanifold of a complex projective space.

It is well known that complex projective space CP^m is a complex m-dimensional (real $2m$) Kaehlerian manifold with constant holomorphic sectional curvature 4. Let CP^m be covered by a system of coordinates neighborhoods $\{U: y^j\}$ and denote by g_{ji} components of the Hermitian metric tensor and by F_i^j those of the complex structure of CP^m, where and in the sequel h, i, j run over the range $\{1, 2, \cdots, 2m\}$. Then we have

$$(2.1)\qquad F_h^i F_j^h = -\delta_j^i,\ \ F_j^h F_i^k g_{hk} = g_{ji},\ \ D_j F_i^h = 0$$

where D_j denotes the operator of the covariant differentiation with respect to g_{ji}.

Let M be an n-dimensional Riemannian manifold cover by a system of coordinate neighborthoods $\{U; x^a\}$ and immersed isometrically in CP^m by the immersion $i; M \to CP^m$ represented by $y^i = y^i(x^a)$. We put $B_a^i = \partial_a y^i$ $(\partial_a = \partial/\partial x^a)$, then B_a^i are tangent to M. Then we have $g_{ba} = g_{ji} B_b^j B_a^i$ since the immersion is isometric, where and in the sequel the indices a, b, c and x, y, z, t run over the range $\{1, 2, \cdots, n\}$ and $\{n+1, \cdots, 2m\}$ respectively.

We denote by N_x^i the unit normals to M and ∇_b the operator of van der Waerden-Bortolotti covariant differentiation with respect to g_{ba}. Then equations of Gauss and Weingarten are respectively given by

(2.2) $\nabla_b B_a^h = A_{ba}^x N_x^h, \quad \nabla_b N_x^h = -A_b{}^a{}_x B_a^h$

where A_{ba}^x are s second fundamental tensors of M in CP^m and $A_b{}^a{}_x = A_{be}^y g^{ac} g_{yx}$, $(g^{ac}) = (g_{ac})^{-1}$, g_{yx} being components of the metric of the normal bundle $T^\perp(M)$

Concerning the transform of B_b^i and N_x^i by F_i^j, we have

(2.3) $F_i^h B_b^i = f_b^a B_a^h + f_b^x N_x^h, \quad F_i^h N_x^i = -f_x^a B_a^h + f_x^y N_y^h$

where we have put $f_{ba} = f_b^c g_{ca}$, $f_{by} = f_b^x g_{xy}$ and $f_{yx} = f_y^z g_{zx}$. Moreover we get

(2.4) $f_{ba} = -f_{ab}, \quad f_{by} = f_{yb}, \quad f_{yx} = -f_{xy}$

Applying F to (2.3) and making use of (2.1), we have

(2.5) $f_a^e f_e^b = -\delta_a^b + f_a^x f_x^b, \quad f_a^e f_e^x + f_a^y f_y^x = 0 = f_x^e f_e^a + f_x^y f_y^a, \quad f_x^z f_z^y + \delta_x^y = f_x^e f_e^y,$

from which, differentiating covariantly along M and using (2.1) and (2, 2), we find

(2.6) $\nabla_c f_b^a = A_c{}^a{}_x f_b^x - A_{cb}^x f_x^a, \quad \nabla_b f_a^x = A_{ba}^z f_z^x - A_{be}^x f_a^e,$

$\nabla_b f_x^a = A_{bx}^e f_e^a - A_b{}^a{}_z f_x^z, \quad \nabla_b f_x^y = A_{be}^y f_x^e - A_b{}^e{}_x f_e^y.$

Since CP^m has constant holomorphic sectional curvature 4, the equations of Gauss, Codazzi and Ricci are respectively given by

(2.7) $K_{dcb}{}^{a} = \delta_d^a g_{cb} - \delta_c^a g_{db} + f_d^a f_{cb} - f_c^a f_{db} - 2 f_{dc} f_b^a + A_d{}^a{}_x A_{cb}^x - A_c{}^a{}_x A_{db}^x,$

(2.8) $\nabla_c A_{ba}^x - \nabla_b A_{ca}^x = f_c^x f_{ba} - f_b^x f_{ca} - 2 f_{cb} f_a^x,$

(2.9) $K_{dcy}{}^{x} = f_d^x f_{cy} - f_c^x f_{dy} - 2 f_{dc} f_y^x + A_{de}^x A_c{}^e{}_y - A_{ce}^x A_d{}^e{}_y,$

where $K_{dcb}{}^{a}$ and $K_{dcy}{}^{x}$ are respectively components of the curvature tensors of M and $T^\perp(M)$. Thus, on the above structure equations, Yano and Kon proved

Theorem 2. 1. (See [7]) *A necessary and sufficient condition for a submanifold of a Kaehlesian manifold M to be CR-submanifold is that the tensor* *fields* f_b^e *and* f_b^x *appearing in* (2.3) *satisfy*

(2.10) $f_e^x f_a^e = 0$ *or equivalently* $f_a^z f_z^y = 0$

where f_a^b *and* f_y^x *are f-structure defined in M and* $T^\perp(M)$ *respectively.*

§ 3. Main results.

Let's denote by $(\bar{M}, M, \pi)$ the compatible submersion with the Hopf-fibration $\tilde{\pi}$ which is given in § 1. In paper [5], Pak proved the equivalence of *CR*-submanifold *M* of CP^m and contact *CR*-submanifold $\bar{M}$ of S^{2m+1} in $(\bar{M}, M, \pi)$. In this point of view, we now assume that $\bar{M}$ is the locally symmetric space as a contact *CR*-submanifold of S^{2m+1}, then *CR*-submanifold *M* satisfies

(3.1) $f_d^e K_{ecba} + f_c^e K_{deba} + f_b^e K_{dcea} + f_a^e K_{dcbe} = 0$

(3.2) $f_d^e (\nabla_c f_{eb} - \nabla_e f_{cb}) + f_c^e (\nabla_e f_{db} - \nabla_d f_{eb}) + f_b^e (\nabla_c f_{de} - \nabla_d f_{ce}) = 0,$

where we have used the equations of co-Gauss and co-Codazzi in $(\bar{M}, M, \pi)$, (See [3], [4]). Hence, substituting (2.6) into (3.2) and making use of (2.10), we have $(f_c^e A_{eb}^x f_y^b) f_{dx} - (f_d^e A_{eb}^x f_y^b) f_{cx} = 0$, from which, transvecting f_z^d and also using (1.5), (1.10), we find

(3.3) $(f_c^e A_{eb}^x f_y^b f_x^w) f_{wz} + f_c^e A_{ebz} f_y^b = 0$

On the other hand, if we assume the *f*-structure f_y^x in the normal bundle of *M* is parallel, then we have $A_{be}^y f_x^e = A_{bx}^e f_e^y$ by (2.6), which, together with (3.3), implies

(3.4) $f_c^e A_{ebz} f_y^b = 0$

from which, transvecting f_a^c, we find by using (2.5)

(3.5) $A_{dbz} f_y^b = P_{xyz} f_d^x$

where we have put $P_{yz}^x = A_{eb}^x f_y^b f_z^e$, and moreover P_{yzx} is symmetric for all indices *x*, *y* and *z* by virture of (2.4) and parallel *f*-structure in $T^\perp(M)$.

We now consider the converse problem. Then we will determine the certain *CR*-submanifolds of CP^m with the above conditions (3.1) and (3.4). Thus, if we use equation (2.7) to (3.1), then we get

$$(f_d^e A_{dex} + f_d^e A_{aex}) A_{cb}^x + (f_b^e A_{cex} + f_c^e A_{ebx}) A_{da}^x - (f_b^e A_{dex} + f_d^e A_{ebx}) A_{ca}^x -$$
$$(f_d^e A_{cex} + f_c^e A_{eax}) A_{db}^x = 0$$

, from which, transvecting f_z^a and using (2.10), (3.4) and (3.5), we have

$$P_{yz}^x f_d^y (A_{cx}^e f_{eb} - f_c^e A_{ebx}) - P_{yz}^x f_c^y (A_{dx}^e f_{eb} - f_d^e A_{ebx}) = 0.$$

Transvecting the above equation with f_w^b and using (2.5), (2.10) and (3.4), we have

(3.6) $\quad P_{yz}{}^{x}(A_{dx}{}^{e}f_{eb}-f_{d}^{e}A_{ebx})f_{w}^{u}f_{u}^{y}+P_{wz}{}^{x}(A_{dx}^{e}f_{eb}-f_{d}^{e}A_{ebx})=0$

On the other side, if we suppose CR-submanifold M has semiflat normal connection, then (2.9) implies $f_{b}^{x}f_{ay}-f_{a}^{x}f_{by}+A_{be}{}^{x}A_{ay}^{e}-A_{ae}{}^{x}A_{by}^{e}=0$, from which, transvecting $f_{z}^{a}f_{v}^{b}$ and taking account of (3.5), we have

(3.7) $\quad P_{zy}{}^{u}P_{vw}{}^{x}f_{e}^{w}f_{u}^{e}-P_{zw}{}^{x}P_{yv}{}^{u}f_{u}^{e}f_{e}^{w}+f_{b}^{x}f_{v}^{b}f_{z}^{a}f_{ay}-f_{z}^{a}f_{a}^{x}f_{by}f_{v}^{b}=0.$

Thus, making use of (3.6), (3.7) and the last equation of (2.5), we find

$$(A_{dv}{}^{e}f_{eb}-f_{d}^{e}A_{ebv})g_{yz}-(A_{dz}{}^{e}f_{eb}-f_{d}^{e}A_{ebz})g_{yv}$$
$$+(A_{dv}^{e}f_{eb}-f_{d}^{e}A_{ebv})f_{z}^{t}f_{ty}-(A_{d}^{e}{}_{z}f_{eb}-f_{d}^{e}A_{ebz})f_{y}^{t}f_{tv}=0,$$

where we have used the equation $A_{cx}{}^{b}f_{z}^{x}=0$ which is proved in [5].

Contracting above equation with respect to y and z and noticing $A_{cx}{}^{b}f_{z}^{x}=0$ implies

$$\{p-(1+\rho)\}(A_{dx}^{e}f_{eb}-f_{d}^{e}A_{ebx})=0,\quad (p=2m-n)$$

where we have put $\rho=\|f_{zt}\|^{2}$, which is locally constant on M since f-structure in $T^{\perp}(M)$ is parallel. Therefore, combining Theorem A and above facts, we have

Theorem 3.1. *Let M be an n-dimensional $(p\neq 1+\rho)$ complete CR-submanifold of CP^{m} with semi-flat normal connection and parallel f-structure in the normal bundle. If the mean curvature vector of M is parallel, and if* (3.1) *and* (3.4) *are valid on M, then M is $CP^{n/2}$ or M is*

$$\pi(S^{m_1}(r_1)\times\cdots\times S^{m_k}(r_k)),\quad n+1=\sum_{1}^{k}m_i,\quad 2\leq k\leq m+1,\quad \sum_{1}^{k}r_i^2=1.$$

When M is generic submanifold of CP^{m}, then we immediately have by (2.8)

Corollary 3.2. *Let M be an n-dimensional $(p>1)$ complete generic submanifold of CP^{m} with flat normal connection. If the mean curvature vector of M is parallel, and if* (2.1) *and* (2.4) *are valid on M, then M is*

$$\pi(S^{m_1}(r_1)\times\cdots\times S^{m_k}(r_k)),\quad n+1=\sum_{1}^{k}m_i,\quad 2\leq k\leq m+1,\quad \sum_{1}^{k}r_i^2=1.$$

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