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Morphisms between Universal Bundles

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Let $V_k(F^n)$ be the *Stiefel variety* of orthogonal k-frames in F^n, and let $G_k(F^n)$ be the *Grassmann variety* of k-dimensional subspaces of F^n, where $F=R$ (reals) or C(complexes). Then $V_k(F^n) \to G_k(F^n)$ with the canonical projection, written ω, is a universal $U_F(k)$ —bundle ([1]), where $U_F(k)$ is the group consisting of all linear transformations $u: F^k \to F^k$ such that $(u(x) \mid u(y)) = (x \mid y)$ for $x, y \in F^k$ and $(x \mid y)$ the inner product on F^k such that

$$x=(x_1, \cdots, x_k), \quad y=(y_1, \cdots, y_k) \quad (x \mid y) = \sum_{i=1}^{k} x_i \bar{y}_i .$$

Milnor gave a construction of a universal $U_k(F)$ —bundle as follows (about 1955). Consider an infinite join

$$E(U_F(k)) = U_F(k) \ * \ U_F(k) \ * \cdots\cdots\cdots$$

Each element of $E(U_F(k))$ is denoted by

$$\langle t, u \rangle = (t_0 u_0, \ t_1 u_1, \cdots, t_k u_k, \cdots),$$

where $t_i \in [0, 1]$ and $u_i \in U_F(k)$ such that only a finite number of $t_i \neq 0$ and $\sum_{0 \leq i} t_i = 1$. In $E(U_F(k))$ we define

$$\langle t, u \rangle = \langle t', u' \rangle \Leftrightarrow t_i = t'_i \text{ for each } i \text{ and } u_i = u'_i \text{ if } t_i = t'_i \neq 0$$

and

$$E(U_F(k)) \times U_F(k) \xrightarrow{\ \mu\ } E(U_F(k))$$

by $\langle t, u \rangle x = (t_0 u_0, \ t_1 u_1, \cdots) x = (t_0 u_0 x, \ t_1 u_1 x, \cdots\cdots)$ for $\langle t, u \rangle \in E(U_F(k))$ and $x \in U_F(k)$. We want to introduce a topology on $E(U_F(k))$ in such a way that $E(U_F(k))$ is a $U_F(k)$ – space.

Define

$$t_i : E(U_F(k)) \to [0, 1] \text{ and } u_i : t_i^{-1}(0, 1] \to U_F(k)$$

as follows. For each $\langle t, u \rangle = (t_0 u_0, \ t_1 u_1, \cdots)$

$$t_i(\langle t, u \rangle) = t_i((t_0 u_0, \ t_1 u_1, \cdots)) = t_i$$

which is the i-th component of t, and for $\langle t, u \rangle \in t_i^{-1}(0, 1]$

$$u_i(\langle t, u \rangle) = u_i((t_0 u_0, \ t_1 u_1, \cdots)) = u_i$$

which is the i-th component of u, It follows that

$$u_i(ax) = u_i(a)x, \quad t_i(ax) = t_i(a)$$

for $a \in E(U_F(k))$ and $x \in U_F(k)$.

The set $E(U_F(k))$ is made into a space by requiring it to have the smallest topology such that the functions t_i and u_i are continuous for all $i = 0, 1, 2, \cdots$. Then by the commutative diagrams

$$\begin{array}{ccc} t_i^{-1}(0, 1] \times U_F(k) & \longrightarrow & t_i^{-1}(0, 1] \\ u_i \times 1 \downarrow & & \downarrow u_i \\ U_F(k) \times U_F(k) & \longrightarrow & U_F(k) \end{array} \qquad \begin{array}{ccc} E(U_F(k)) \times U_F(k) & \longrightarrow & E(U_F(k)) \\ P_1 \downarrow & & \downarrow t_i \\ E(U_F(k)) & \longrightarrow & [0, 1] \end{array}$$

μ is continuous, where $\cdot$ is the product operator of $U_F(k)$ and p_1 is the projection on the first argument. Therefore $E(U_F(k))$ is a $U_F(k)$-space. We denote the quotient space $E(U_F(k)) \bmod U_F(F)$ by $B(U_F(k))$. Then $E(U_F(k)) \rightarrow B(U_F(k))$ with the canonical projection, written $\omega(U_F(k))$, is a universal $U_F(k)$-bundle ([1]).

In this paper, we shall make a $U_F(k)$-bundle morphism from ω to $\omega(U_F(k))$ (Theorem 5) and prove its property (Proposition 6)

As is well known, $G_k(F^n)$ is a finite CW-complex ([2]), and thus it is compact. Moreover, the direct limit $G_K(F^\infty)$ of the sequence $G_k(F^n) \subset G_K(F^{n+1}) \cdots$ of compact space is paracompact ([2]).

We know that the projection $p: V_k(F^n) \rightarrow G_k(F^n)$ is locally trivjal. For each subset of k elements $H \subset \{1, 2, \cdots, n\}$, we define O_H to be the open set $O(F_H)$, where

(i) $F_H = \sum_{i \in H} F_{e_i} \cong F^k \subset F^n$ $((e_1, \cdots e_n)$ is the usual orthonormal base of $F^n)$,

(ii) $O(F_H) = \{W \in G_k(F^n) \mid W$ is not orthogonal to $F_H\}$.

Then O_H is a locally trivialization domain ([1]), *i. e.*, there exists a homeomorphion $h_H : O_H \times U_F(k) \rightarrow p^{-1}(O_H)$ such that the diagram

$$\begin{array}{ccc} O_H \times U_F(k) & \xrightarrow{\quad h_H \quad} & p^{-1}(O_H) \\ & {\scriptstyle p_1} \searrow \quad \swarrow {\scriptstyle p} & \\ & O_H & \end{array}$$

is commutative, and

(iii) For each point $W \in O_H$, $h_H \mid W \times U_F(k)$ is a $U_F(k)$-map, *i. e.*,

$$h_H(W \times xy) = (h_H(W \times x))\, y$$

for $x, y \in U_F(k)$.

Definition 1. An open covering $\{U_i\}_{i \in s}$ of a topological space B is ***numer***-

able if *there* exists a locally finite partition of unity $\{\psi_\iota\}_{\iota\in S}$ such that $\varphi_\iota^{-1}(0, 1]\subset U_\iota$ for each $\iota\in S$. A principal G-bundles ξ over a space B is ***numerable*** provided a numerable cover $\{U_\iota\}_{\iota\in S}$ of B such that $\xi\,|\,U_\iota$ is trivial for all $i\in S$, where G is a topological group.

Proposition 2. $p: V_K(F^n) \to G_K(F^n)$ is a numerable $U_F(k)$-bundle.

Proof) By the description above $\{O_H\}_{H\subset\{1,2,\cdots,n\}}$ is a finite open covering of $G_K(F^n)$ such that $V_K(F^n)\,|\,O_H$ is trivial. Hence there is a partition of unity $\{\varphi_H\}_{H\subset\{1,2,\cdots,n\}}$ such that $\varphi_H^{-1}(0, 1]\subset O_H$. This implies that $p: V_K(F^n) \to G_K(F^n)$ is numerable. *q. e. d.*

Lemma 3. There is a $U_F(k)$-map from $V_K(F^{mk})$ to the join

$$E(U_F(k))(\mathrm{mk})=U_F(k) \ast\cdots\ast U_F(k) \quad ({}_{mk}C_k\text{-times}).$$

Proof) We order $\{O_H\}_{H\subset\{1,\cdots,mk}$ as $\{O_0,\cdots,O_q\}$ $(q={}_{mk}C_k-1)$. Let $h_n: O_n \times U_F(k) \to V_K(F^{mk})\,|\,O_n$ be an isomorphism defining the locally trivial character of $p: V_K(F^{mk})\to G_K(F^{mk})$, and let $\{\varphi_i\}_{i=0,\cdots q}$ be a partition of unity of $\{O_i\}$ $i=0,\cdots q$. We define $g: V_K(F^{mk})\to E(U_F(k))(mk)$ by the relation

$$g(z)=(\varphi_0(p(z))\; p_0(h_0^{-1}(z)),\ \cdots,\ \varphi\,(p(z))\; p_q(h_q^{-1}(z))),$$

where $z\in V_K(F^{mk})$ and $p_n: O_n\times U_F(k)\to U_F(k)$ is the projection on the second argument. We have to note that if $h_n^{-1}(z)$ is undefined then $\varphi_n(p(z))=0$. By our definition and the condition (iii) above (*i. e,* $h_n(za)=h_n^{-1}(z)a$ for $z\in V_K(F^{mk})$ and $a\in U_F(k)$), for each $a\in U_F(k)$ $g(za)=g(z)a$, and thus g is well defined(Note that $\sum_{i=0}^{q}\varphi_i(p(z))=1$) . Since g is continuous, g is a reguired $U_F(k)$-map.
q. e. d.

Corollary 4. Let $B(U_F(k))(mk)$ be the quotient $E(U_F(k))(mk) \bmod U_F(k)$. Then there is a $U_F(k)$-bundle morphism (g, f) satisfying the commutative diagram

$$\begin{array}{ccc}
V_K(F^{mk}) & \xrightarrow{\quad g \quad} & E(U_F(k))(mk) \\
p\,\big\downarrow & & \big\downarrow\,\widetilde{p} \\
G_K(F^{mk}) & \xrightarrow{\quad f \quad} & B(U_F(k))(mk),
\end{array}$$

where $\widetilde{p}$ is the bundle projection.

Proof) It suffices to define a continuous function f. For each $W \in G_k(F^{mk})$ we define such that

$$f(W) = \widetilde{p} \cdot g \cdot p^{-1}(W).$$

Then, since p and $\widetilde{p}$ are projections and g is continuous, f is continuous. Also, $p \cdot f = \widetilde{p} \cdot g$ hold. *q. e. d.*

Theorem 5. There is a $U_F(k)$-bundle morphiom $(g, f) : \omega \to \omega(U_F(k))$.

Proof) By the description above $G_k(F^\infty)$ is a paracompact space. Since $p : V_k(F^\infty) \to G_k(F^\infty)$ is a locally trivial principal $U_F(k)$-bundle $\omega = (V_k(F^\infty), p, G_k(F^\infty))$ is numerable ([1]). Therefore there exists a countable partition of unity $\{\varphi_i\}_{0 \leq i}$ such that $V_k(F^\infty) \mid \varphi_i^{-1}(0, 1]$ is trivial for all $i = 0, 1, 2, \cdots$ ([1]).

$$g : V_k(F^\infty) \to E(U_F(k)) \text{ is defined by}$$

$$g(z) = (\varphi_0(p(z))\ p_0(h_0^{-1}(z)), \cdots, \varphi_k(p(z))\ p_k(h_k^{-1}(z), \cdots),$$

where $O_n = \varphi_n^{-1}(0, 1]$, $h_n : O_n \times U_F(k) \to p^{-1}(O_n)$ is an isomorphism defining the locally trivial character of ω and $p_n : O_n \times U_F(k) \to U_F(k)$ is the projection on the second argument. As in the proof of Lemma 3 g is well defined. By the commutative diagram

$$\begin{array}{ccc} V_k(F^\infty) & \xrightarrow{\quad g \quad} & E(U_F(k)) \\ p \downarrow & & \downarrow \widetilde{p} \\ G_k(F^\infty) & \xrightarrow{\quad f \quad} & B(U_F(k)) \end{array}$$

f is well defined, where $\widetilde{p}$ is the bundle projection. *q. e. d.*

Proposition 6. If $(g_1, f_1), (g_2, f_2) : \omega \to \omega(U_F(k))$ are $U_F(k)$-bundle morphisms, f_1 and f_2 are homotopic.

Proof) Let β be the category consisting of all $U_F(k)$-bundles over $G_k(F^\infty)$ and all $U_F(k)$-bundle morphisms. Then every morphism in β is an isomorphism ([1]). Therefore we have

$$f_1^*(\omega(U_F(k)) \cong V_k(F^\infty) \cong f^*(\omega(U_F(k)).$$

Since $\omega(U_F(k)$ is universal, we have $f_1 \simeq f_2$. *q. e. d.*

References

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