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# Automorphisms on Cuntz algerbras

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In 1977, J. Cuntz(3) investigated the $C^*$-algebra $C^*(S_1,\cdots,S_n)$ generated by isometries $S_1,\cdots,S_n$ satisfying $\sum_{i=1}^{n} S_iS_i^*=1$ ($\sum_{i=1}^{r} S_iS_i^*<1$ for all $r\in N$ if $n=\infty$). He showed that the isomorphism class of $C^*(S_1,\cdots,S_n)$ does not depend on the choice of $S_1,\cdots,S_n$, that is, if $T_1,\cdots T_n$ is a second family of isometries with $\sum_{i=1}^{n} T_iT_i^*=1$ ($\sum_{i=1}^{r} T_iT_i^*<1$ for all $r\in N$ if $n=\infty$), then $C^*(T_1,\cdots,T_n)$ is canonically isomorphic to $C^*(S_1,\cdots,S_n)$. In the following (the isomorphism class of) $C^*(S_1,\cdots,S_n)$ is denoted by $O_n$.

Recenty R. J. Archbold (1) studied an automorphism $\theta$ of $O_2=C^*(S_1, S_2)$ satisfying $\theta(S_1)=S_2$, $\theta(S_2)=S_1$. He called $\theta$ the flip-flop automorphism on $O_2$ and established the following theorem.

**THEOREM** The flip-flop automorphism $\theta$ on the Cuntz algebra $O_2$ is outer.

Since the flip-flop automorphism is induced by a permutation on $\{S_1, S_2\}$, it is natural to ask whether Theorem is able to generalize to the case of permutations on $\{S_1,\cdots,S_n\}$.

Let $S(n)$ be the symmetric group of degree $n$.

**THEOREM 1** The symmetric group $S(n)$ has a representation as a subgroup of outer automorphisms on the Cuntz algebra $O_n$ for $n=2, 3\cdots$.

Let $T(n)$ be the n-dimensional torus group for $n=2, 3,\cdots$. Following after J. Phillips (5) and M. Choda (2), we define an automorphism $\alpha_r$ on the Cuntz algebra $O_n$ by $\alpha_r(S_i)=r_iS_i$ ($i=1, 2,\cdots,n$), for $r=(r_1,\cdots,r_n)\in T(n)$.

**THEOREM 2** The torus group $T(n)$ has a continuous representation as a subgroup of outer automorphisms on the Cuntz algebra $O_n$ for $n=2, 3,\cdots$.

To prove Theorem 1 and 2, we shall show that the group $U(n)$ of all $n\times n$ unitary matrices is faithfully represented as a subgroup of outer automorphism on the Cuntz algebra $O_n$. Similarly, separable compact groups and countable discrete groups are faithfully represented as a subgroup of outer automorphism on $O_\infty$.

In this paper we shall consider a topology of pointwise norm convergence on the group Aut $O_n$ of all automorphisms on the Cuntz algebra $O_n$ for $n=2, 3, \cdots, \infty$.

Let $M_n$ be the algebra of all $n\times n$ matrices. For $a=(a_{ij})\in M_n$ we define $\alpha_a(S_j)=\sum_{i=1}^{n} a_{ij} S_i$. Then we have $\alpha_{ab}=\alpha_a\alpha_b$ on the linear span of $S_1,\cdots, S_n$ for $a, b\in M_n$.

**Lemma 3** For $u=(u_{ij})$ in $M_n$, $\alpha_u$ can be extended to an automorphism on the Cuntz algebra $O_n$ if and only if $u$ is in $U(n)$.

**Proof** Let $u=(u_{ij})\in U(n)$. Put $T_i=\alpha_u(S_i)$, then

$$T_i^*T_i=(\sum_{k=1}^{n} u_{ki} S_k)^* (\sum_{k=1}^{n} u_{ki} S_k)=\sum_{k=1}^{n} \bar{u}_{ki} u_{ki}=(u^*u)_{ii}=1.$$

Hence $T_i$ is an isomery for $i=1, 2, \cdots$, n.

Moreover we have $\sum_{i=1}^{n} T_iT_i^*=\sum_{i=1}^{n} (\sum_{j=1}^{n} u_{ji} S_j) (\sum_{k=1}^{n} u_{ki} S_k)^*=\sum_{i,j,k=1}^{n} u_{ji}\bar{u}_{ki} S_jS_k^*$

$$=\sum_{j,k=1}^{n} \delta_{kj} S_jS_k^*=\sum_{j=1}^{n} S_jS_j^*=1.$$

since $C^*(T_1,\cdots, T_n)=C^*(S_1,\cdots, S_n)$, $\alpha_u$ is an automorphism on $O_n$. Conversely, if $\alpha_u$ can be extended to an automorphism on $O_n$, then

$$\delta_{ij}=\alpha_u(S_i^*S_j)=\alpha_u(S_i)^*\alpha_u(S_j)=\sum_{h,k=1}^{n} \bar{u}_{hi}u_{kj}S_h^*S_k$$

$$=\sum_{k=1}^{n} \bar{u}_{ki}u_{kj}=(u^*u)_{ij}.$$

Hence $u$ is in $U(n)$. Q. E. D.

Let $U(\infty)$ be the union of $U(n)$ ; $n=1, 2, \cdots$.

The proof of the following theorem is inspired by R. J. Archbold (1 : Theorem 1)

**THEOREM 4** The unitary group $U(n)$ has a faithful continuous representation as a subgroup of outer automorphisms on the Cuntz algebra $O_n$ for $n=2, 3, \cdots, \infty$.

**Proof** Let $u, v\in U(n)$. Then $\alpha_{uv}=\alpha_u\alpha_v$ on the linear span of $S_1, S_2\cdots$, $S_n$ Since $\alpha_{uv}$, $\alpha_u$, $\alpha_v$ are automorphisms and $O_n$ is generated by $S_1,\cdots, S_n$, it followes that $\alpha_{uv}=\alpha_u\alpha_v$ on $O_n$.

Next we shall show that $\alpha_u$ is an outer automorphism on $O_n$ for $u\in U(n)$ in the case of $n<\infty$. For $u\in U(n)$, there exist $v$ and $w$ in $U(n)$ such that $u=$

$vwv^*$ and $w$ is a diagonal matrix with eigenvalues $(\lambda_1, \cdots, \lambda_n)$. Then $\alpha_u$ is outer if and only if $\alpha_w$ is outer. Therefore it is sufficient to show that $\alpha_u$ is outer for a diagonal matrix $u \in U(n)$ with eigenvalues $(\lambda_1, \cdots, \lambda_n)$.

We may assume that $\lambda_1 \neq 1$ without loss of generality. Suppose that $\alpha_u$ is inner. Then $\alpha_u = Ad\ V$ for some unitary $V \in O_n = C^*(S_1, \cdots, S_n)$, where $(Ad\ V)\ (X) = VXV^*$ for all $X \in O_n$.

Let $H$ be a separable Hilbert spase with complete orthonormal basis $\{e_k; k = 1, 2, \cdots\}$. We realize $S_i$ on $H$ by specifying that $S_i e_k = e_{nk-(n-i)}$. Then $S_1 e_1 = e_1$ and $Ve_1 = \sum_{k=1}^{\infty} x_k e_k \neq 0$ for some $(x_k) \in 1^2(N)$. Hence

$\alpha_u(S_1)\ (Ve_1) = (VS_1V^*)\ (Ve_1) = VS_1 e_1 = Ve_1 = \sum_{k=1}^{\infty} x_k e_k.$

On the other hand $\alpha_u(S_1)\ (Ve_1) = \lambda_1 S_1 (\sum_{k=1}^{\infty} x_k e_k) = \sum_{k=1}^{\infty} \lambda_1 x_k e_{nk-(n-1)}$. Comparing the Fourier coefficient, we have $x_k = 0$. This is a contradiction. Therefore $\alpha_u$ is outer. Secondly we shall show that $\alpha_u$ is an outer automorphism on $O_\infty$ for $u \in U(\infty)$. Put

$$S_i e_k = e_h \text{ where } h = 2^{i-1}(2k-1).$$

By the same argument as the above case, we can conclude that $\alpha_u$ is an outer automorphism on $O_\infty$ for $u \in U(\infty)$. We shall show the faithfullness of the representation $\alpha$ of $U(n)$. If $\alpha_u = 1$ for $u = (u_{ij}) \in U(n)$, then $S_j = \alpha_u(S_j) = \sum_{k=1}^{n} u_{kj} S_k$ and $\delta_{ij} = S_i^* S_j = S_i^* \alpha_u(S_j) = \sum_{k=1}^{n} u_{kj} S_i^* S_k = u_{ij}$. Hence $u = 1$. Therefore the representation $\alpha$ is faithful. We shall show that $\alpha$ is continuous. Let $u(k) \to u$ $k \to \infty)$ in $U(n)$, then $(u(k))_{ij} \to u_{ij}$ $(k \to \infty)$. It follows that $\| \alpha_{u(k)}(x) - \alpha_u(x) \| \to 0$ $(k \to \infty)$ for all $x \in P$, where $P$ is the dense$^*$-algebra generated by $S_1, \cdots, S_n$. Since $\alpha_{u(k)}$ is isometric for all $k = 1, 2, \cdots$, $\| \alpha_{u(k)}(X) - \alpha_u(X) \| \to 0$ $(k \to \infty)$ for all $x \in O_n$. Hence $\alpha$ is continuous.

**THEOREM 5** (1) Any separable compact group has a faithful continuous representation as a subgroup of outer automorphisms of the Cuntz algebra $O_\infty$.

(2) Any countable discrete group has a faithful representation as a subgroup of outer automorphisms of the Cuntz algebra $O_\infty$.

**Proof**

(1) Let $G$ be a separable compact group and $\pi$ the left regular representation.

Then there are finite dimensional unitary representations $\pi_i\,(i=1, 2, \cdots)$ such that

$$\pi=\sum_{i=1}^{n}\oplus\pi_i \quad (dim\ \pi_i<\infty)\ (4\,;15.1.3).\quad \text{Put } n(i)=dim\,\pi_i.$$

We can assume that $O_\infty$ is the C*-algebra generated by isometries $T_1^{(1)}$, $T_2^{(1)}$ $\cdots$, $T_{n(1)}^{(1)}$, $T_1^{(2)}, \cdots, T_{n(2)}^{(2)}, \cdots$, $T_k^{(i)}, \cdots$, $(1\leqq k\leqq n(i)$ for $i=1, 2, \cdots)$, on a Hilbert space $H$ with orthogonal range projections whose finte sum is less than 1.

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$$\pi_i(g)=(u_{mk}^{(i)}(g))$$

be the $n(i)\times n(i)$ matrix representation of unitary $\pi_i(g)$ for $g\in G$.

We define the map $\alpha$ from $G$ to the group Aut $O_\infty$ of all automorphism on $O_\infty$ by

$$S_k^{(i)}=\alpha_g(T_k^{(i)})=\sum_{m=1}^{n(i)}u_{mk}^{(i)}(g)\ \ T_m^{(i)} \quad 1\leqq k\leqq n(i).$$

Since $O_\infty$ is also generated by isometries

$$S_1^{(1)}, S_2^{(1)}, \cdots, S_{n(1)}^{(1)},\ \ S_1^{(2)}, \cdots, S_{n(2)}^{(2)}, \cdots, S_k^{(i)}, \cdots$$

$(1\leqq k\leqq n(i)$ for $i=1, 2, \cdots)$, with orthogonal range projections whose finite sum is less than 1, $\alpha_g$ is an automorphism of $O_\infty$ for $g\in G$.

A similar method used in the proof of theorem 4 establishes that $\alpha_g$ is outer for $g\neq 1$. It is clear that $\alpha$ is a group monomorphism. We shall show that the representation $\alpha$ is continuous. Let $g(n)\to g$, $(n\to\infty)$ in $G$. Then $u_{mk}^{(i)}(g(n))\to u_{mk}^{(i)}(g)$ $(n\to\infty)$ because $\pi$ is strongly continuous.

It follows that $\|\alpha_{g(n)}(x)-\alpha_g(x)\|\to 0$ for all $x\in P$, where $P$ is the dense *-algebra generated by $\{T_k^i\ ;\ 1\leqq k\leqq n(i), i=1, 2, \cdots\}$.

Since $\alpha_g$ is isometric for $g\in G$, we have

$\|\alpha_{g(n)}(x)-\alpha_g(x)\|\to 0$ for all $x\in O_\infty$. Hence $\alpha$ is continuous.

(2) $G$ : countable discrete group. We may assume that countably infinite ; $G=\{g_1, g_2, \cdots\}$. For each fixed $s\in G$, there is $g_m\in G$ such that $sg_n=g_{(m)}$ for all $g_n\in G$. the correspondence $g_n\to g_m$ induces a permutation $1_s$ on $N$ such that $1_s(n)=m$. Then 1 is a group monomorphism from $G$ to the permutation group on $N$.

Let $S_1, S_2, \cdots$, be isometries with orthogonal range projections whose finite sum is less than 1. Then $O_\infty=C^*(S_1, S_2, \cdots,)$. For $s\in G$, define an automorphism $\alpha_s$ on $O_\infty$ by $\alpha_s(S_n)=S_{1(n)}$

where $1 = 1_s$ the permutation induced by $s \in G$. $\alpha_s$ is well defined because $S_{1}(n)(n=1, 2, \cdots,)$ are also isometries with orthogonal range projections whose finite sum is less than 1 and $C^*(S_1, S_2, \cdots) = C^*(S_{1(1)}, S_{1(2)}, \cdots)$.

It is clear that the map $s \rightarrow \alpha_s$ from $G$ to Aut $O_\infty$, the automorphism group of $O_\infty$, is a group monomorphism. We shall show that for $s \neq 1$, $\alpha_s$ is outer. We may assume that $\alpha_s(S_1) = S_2$ for $s \in G$. We can realize $O_\infty$ on a separáble Hilbert space $H$ with orthonormal basis $e_1, e_2, \cdots$ such that $S_i e_n = e_h$ where $h = 2^{i-1}(2n-1)$ $(i, n = 1, 2, \cdots)$. Suppose that $\alpha_s$ is inner. Then there exists a unitary $U \in O_\infty$ such that $US_1U^* = S_2$. Put $x = Ue_1 \neq 0$. Then $S_2x = S_2Ue_1 = US_1e_1 = Ue_1 = x$ because $S_1e_1 = e_1$. However a simple calculation as the proof of THEOREM 4 shows that $x = 0$. This contradiction shows that $\alpha$ cannot be inner. Q. E. D.

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