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ON CHARACTERIZATION OF THE CLASS OF CLOSE-TO-CONVEX ANALYTIC FUNCTIONS

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§ 1. INTRODUCTION

W. Kaplan [7] has called an analytic function $f(z)$, ***close-to-convex*** in $|z|<1$ if there exists an analytic function $\phi(z)$, convex and schlicht in $|z|<1$, such that

(1.1) $$Re\left[\frac{f'(z)}{\phi'(z)}\right]>0 \quad \text{for } |z|<1.$$

Here a function is called convex if it maps the unit disk $|z|<1$ conformally onto a convex region. The necessary and sufficient condition for $\phi(z)$ to be a convex function is that

(1.2) $$Re\left[1+\frac{z\phi''(z)}{\phi'(z)}\right]>0 \quad \text{for } |z|<1.$$

Since $F(z)=z\phi'(z)$ is starlike with respect to the origin and schlicht in $|z|<1$, (1.1) may be written in the alternative form

(1.3) $$Re\frac{zf'(z)}{F(z)}>0 \quad \text{for } |z|<1.$$

In this case $f(z)$ is said to be ***close-to-convex in*** $|z|<1$ ***relative to the convex function***

(1.4) $$\phi(z)=\int_0^z \frac{F(t)}{t}\,dt$$

where $F(z)$ is analytic and normalized so that $F(0)=0$, $F'(0)=1$ satisfying $Re[zF'(z)/F(z)]>0$, $|z|<1$. Kaplan has shown [7] that every close-to-convex function $f(z)$ is schlicht in $|z|<1$.

In this paper, we will characterize the close-to-convex functions intrinsically, without reference to a convex function $\phi(z)$. One of such characterization is obtained as follows: $f(z)$ is close-to-convex in $|z|<1$ if and only if $\int_{\theta_1} Re\left[1+z\frac{f''(z)}{f'(z)}\right]d\theta>-\pi$ when $\theta_1<\theta_2$, $z=re^{i\theta}$ and $r<1$. In connection with this characterization, we extend certain results due to Pommerenke [14], Robertson [22], Miller, Mocanu, and Reade [11] in section III obtaining subclasses of close-to-convex functions.

We shall say that $f(z)=\sum_1^\infty a_n z^n$, analytic for $|z|<1$, is of class K_α, $0\leq\alpha\leq 1$, if and only if $f(z)$ satisfies (i) $f'(z)\neq 0$, (ii) for all $\theta_1<\theta_2$ and for all r, $0\leq r<1$, $\int_{\theta_1}^{\theta_2} Re[1+z\frac{f''(z)}{f'(z)}]\,d\theta>-\pi\alpha, (z=re^{i\theta})$ holds. We note that K_0 consists of convex functions, K_1 consists of close-to -convex functions. For functions of class K_α, we obtained the following result: If $f(z)=\sum_1^\infty a_n z^n\in K_\alpha$, then $f(z)$ is univalent and

$$|a_n|\leq[1+\alpha(n-1)]|a_1|,\quad n=2,\ 3,\cdots.$$

Moreover, we introduce a class of functions K_α^* which bear much the same relation to star maps that the close-to-convex functions bear to convex maps. We shall say that $F(z)=\sum_1^\infty b_n z^n$, analytic for $|z|<1$, is in class K_α^*, $0\leq\alpha\leq 1$, if and only if $F(z)$ satisfies the following conditions; (i) $F'(0)\neq 0$, (ii) $F(z)=0$ if and only if $z=0$, and (iii) the inequality

$$(1.5)\qquad \int_{\theta_1}^{\theta_2} Re[zF'(z)/F(z)]\,d\theta>-\pi\alpha,\quad z=re^{i\theta},$$

holds for all $\theta_1<\theta_2$ and for all r, $0<r<1$. We see that K_0^* consists of all schlicht maps which are star with respect to $F(0)=0$, and K_1^* consists of maps so called "close-to-star." For functions of class K_α^* we have the following; A necessary and sufficient condition that $F(z)\in K_\alpha^*$, $0<\alpha\leq 1$, is that there exist a univalent map $\sigma(z)=\sum_1^\infty c_n z^n$, star with respect to $\sigma(0)=0$, such that $|\arg F(z)/\sigma(z)|<\frac{\pi\alpha}{2}$. Also we proved that if $F(z)=\sum_1^\infty b_n z^n\in K_\alpha^*$, then $|b_n|\leq n[1+\alpha(n-1)]|b_1|$, $n=2,3,\cdots$.

Due to Pommerenke's lemma [15], we obtain another coefficient inequality for the class K_α. *i.e.*, let $f(z)=\sum_1^\infty a_n z^n$ belong to K_α with $0\leq\alpha<1$ or let $f(z)$ be convex in one direction. If $M(r)=\max_{|z|=r}|f(z)|\leq\frac{B}{(1-r)^\lambda}$, $(0\leq r<1)$ with $\lambda\geq 0$, then $|a_n|\leq JBn^{\lambda-1}$, $(n=1,\ 2,\cdots)$. where J depends only on α in the first case and $J\leq 16/\pi$ in the second case.

In addition, Miller, Mocanu and Reade [11] introduced the class of α-convex functions. He called a function $f(z)$ an α-convex function if $f(z)f'(z)/z \neq 0$ for $|z|<1$ and if for some nonnegative real number α,

$$Re\left[(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right]>0, \quad |z|<1.$$

Modifying this idea to close-to-convex functions, we shall introduce new subclass of close-to-convex functions. Let $f(z)=z+a_2z^2+\cdots$, be analytic for $|z|<1$ satisfying the conditions (i) $f(z)f'(z)/z\neq 0$ for $|z|<1$, and (ii) for some nonnegative real number α and for some starlike function $\phi(z)=z+\cdots$,

$$(1.6) \qquad Re\left[(1-\alpha)\frac{zf'(z)}{\phi(z)}+\alpha\frac{[zf'(z)]'}{\phi'(z)}\right]>0, \quad |z|<1.$$

For a fixed α, we denote this class of functions by M_α, and we show that if $f(z)\in M_\alpha$, then $f(z)$ is close-to-convex function in $|z|<1$ by using the subordinate principles [20] and Jack's lemma [2] and also we derive an integral representation formula for constructing this class of functions.

In section IV, we extend the concept of close-to-convex functions to meromorphic functions

$$(1.7) \qquad f(z)=\frac{1}{z}+a_0+a_1z+\cdots+a_nz^n+\cdots$$

regular in $0<|z|<1$ and with a simple pole at the origin. We say that $f(z)$, when given by (1.7), is close-to-convex in the punctured circle $0<|z|<1$ relative to $F(z)$ if there exists a meromorphic, rtarlike, schlicht function $F(z)$ in $|z|<1$ with a simple pole at the origin, given by

$$(1.8) \qquad F(z)=\frac{b_{-1}}{z}+b_0+b_1z+\cdots+b_nz^n+\cdots \quad (b_{-1}\neq 0)$$

such that

$$(1.9) \qquad Re[zf'(z)/F(z)]>0 \text{ for } |z|<1.$$

By a modification of Kaplan's argument [7], we show that in the meromorphi case (1.9) is equivalent to

$$(1.10) \qquad \int_{\theta_1}^{\theta_2} Re\left[1+re^{i\theta}\frac{f''(re^{i\theta})}{f'(re^{i\theta})}\right]d\theta<\pi$$

for $\theta_1<\theta_2$, $0<r<1$, if $f(z)$ if $f(z)$ is analytic with a nonvanishing derivative in $0<|z|<1$ and with a simple pole at $z=0$.

Finally, in section V, we discussed an open question raised by Robertson [21] concerning the coefficients of close-to-convex functions. He pointed out in 1968 that he was not sure whether the inequality

$$(1.11) \qquad \big|\,n|a_n|-m|a_m|\,\big|\leq|n^2-m^2|$$

is true for all $f(z)=z+\sum_{2}^{\infty} a_n z^n$ that are close-to-convex functions if $n-m$ is not even. Here we prove that the inequality (1.11) holds for all positive integers n and m by using the inequalities due to MacGregor [5], Lebedev and Milin [5].

II. CHARACTERIZATION OF CLOSE-TO-CONVEX FUNCTIONS

DEFINITION. Let $f(z)$ be analytic for $|z|<1$. Then $f(z)$ is close-to-convex for $|z|<1$ if there exists a function $\phi(z)$, convex and schlicht for $|z|<1$, such that $f'(z)/\phi'(z)$ has positive real part for $|z|<1$.

The class of close-to-convex functions clearly includes the convex functions themselves, as well as the functions $f(z)$ whose derivative has positive real part in the unit disk. The normalized schlicht functions $f(z)$ which map $|z|<1$ onto a domain starshaped with respect to the origin are characterized by the inequality

$$(2.1) \qquad Re\left(\frac{zf'(z)}{f(z)}\right)>0, \quad |z|<1.$$

Since $\phi(z)=\int_0^z f(z)/z\, dz$ is known to be convex, it follows that the star mappings are included in the close-to-convex functions.

If $h(z)=\log \phi'(z)$ is chosen to be analytic, then the condition that $\phi(z)$ be convex is expressed by the inequality

$$(2.2) \qquad Re[1+zh'(z)]>0.$$

Accordingly, if $f(z)$ is analytic for $|z|<1$, then $f(z)$ is close-to-convex for $|z|<1$ if and only if there exists a function $h(z)$, analytic for $|z|<1$, such that

$$(2.3) \qquad Re[f'(z)\, e^{-h(z)}]>0, \qquad Re[1+zh'(z)]>0.$$

From the familiar integral representation of a function with positive real part, [12] we obtain the expressions

$$(2.4) \qquad f'(z)=e^{h(z)}\left[\int_0^{2\pi}\frac{e^{i\theta}}{e^{i\theta}-z}du(\theta)+i\alpha\right],$$

$$(2.5) \qquad h'(z)=\frac{1}{z}\left[\int_0^{z}\frac{e^{i\theta}+z}{e^{i\theta}-z}dv(\theta)+i\beta-1\right],$$

from which an integral represention for $f(z)$ in terms of two monotone non-decreasing functions $u(\theta)$ and $v(\theta)$ can be obtained.

LEMMA 1. Every close-to-convex function is schlicht [7].

Proof) It is known [12] that if $g(z)$ is analytic in a convex domain D and

(2.6) $$Re[g'(z)]>0 \quad \text{in } D,$$

then $g(z)$ is schlicht in D. If $\phi(z)$ is a schlicht map of $|z|<1$ onto D, then $f(z)=g[\phi(z)]$ is also schlicht. Since $f'(z)=g'(\phi)\,\phi'(z)$, $f(z)$ satisfies the condition

(2.7) $$Re\left[\frac{f'(z)}{\phi'(z)}\right]>0 \quad \text{for } |z|<1.$$

Concersely, if $f(z)$ satisfies (2.7), then $g(z)=f[\phi^{-1}(z)]$ satisfies (2.6) and $f(z)=g[(\phi(z)]$ is schlicht for $|z|<1$.

LEMMA 2. Let $t(\theta)$ be a real function of θ for $-\infty<\theta<\infty$ such that

(2.8) $$t(\theta+2\pi)-t(\theta)=2\pi,$$

(2.9) $$t(\theta_1)-t(\theta_2)<\pi \quad \text{for} \quad \theta_1<\theta_2.$$

Then there exists a real function $s(\theta)$ which is monotonic non-decreasing and satisfies the conditions

(2.10) $$s(\theta+2\pi)-s(\theta)=2\pi,$$

(2.11) $$|s(\theta)-t(\theta)|\leq\frac{1}{2}\pi.$$

Proof) Let

$$s(\theta)=\underset{\theta'\leq\theta}{l.u.b.}\ t(\theta')-\frac{1}{2}\pi.$$

Then $s(\theta)$ is non-decreasing. By (2.9), $t(\theta')$ is bounded above, for $\theta'<\theta$, by $t(\theta)+\pi$. Hence the least upper bound is finite and

$$(\theta)\leq t(\theta)+\frac{1}{2}\pi.$$

Furthermore, since $t(\theta)\leq l.u.b.\ t(\theta')$ for $\theta'\leq\theta$,

$$s(\theta)\geq t(\theta)-\frac{1}{2}\pi.$$

Hence (2.11) is proved; (2.10) follows from (2.8). Thus the lemma 2 is established.

THEOREM 2.1. A necessary and sufficient condition that a function $f(z)$, analytic and with $f'(z)\neq0$ for $|z|<1$, be close-to-convex is that

(2.12) $$\int_{\theta_1}^{\theta_2} Re\left[1+re^{i\theta}\frac{f''(re^{i\theta})}{f'(re^{i\theta})}\right]d\theta>-\pi$$

holds for $\theta_1<\theta_2$ and $r<1$.

Proof) (Necessity). Let $\phi(z)$ be a convex schlicht function for $|z| < 1$ and let $p(z)=\arg f'(z)$, $q(z)=\arg \phi'(z)$ be chosen to be continuous for $|z| < 1$. Since $f'(z)$ and $\phi'(z)$ have no roots for $|z|<1$, such a choice is possible. Since $Re[f'(z)/\phi'(z)]>0$ for $|z|<1$, at each z

$$|p(z)-q(z)+2k\pi|<\frac{1}{2}\pi$$

for some $k=0, \pm 1, \cdots$. Because of the continuity of $p(z)$ and $q(z)$, k must be independent of z. If $p(z)$ is properly chosen, $k\equiv 0$, and it will be assumed that such a choice has been made, so that

(2.13) $$|p(z)-q(z)|<\frac{1}{2}\pi \quad \text{for} \quad |z|<1.$$

We now introduce the functions

(2.14) $$P(r, \theta)=p(re^{i\theta})+\theta, \quad Q(r, \theta)=q(re^{i\theta})+\theta,$$

which are defined for $0\leq r<1$ and all real θ. Condition (2.13) becomes

(2.15) $$|P(r, \theta)-Q(r, \theta)|<\frac{1}{2}\pi$$

The condition that $\phi(z)$ be a convex mapping is described by (2.2) or equivalently by the condition

(2.16) $$\frac{\partial Q}{\partial \theta}>0 \ :$$

Thus $Q(r, \theta)$ is monotone increasing in θ for each fixed r. Now, if $\theta_1<\theta_2$,

$$\begin{aligned} &P(r, \theta_1)-P(r, \theta_2)\\ &=[P(r, \theta_1)-Q(r, \theta_1)]-[P(r, \theta_2)-Q(r, \theta_2)]\\ &\quad+[Q(r, \theta_1)-Q(r, \theta_2)]<[P(r, \theta_1)-Q(r, \theta_1)]\\ &\quad-[P(r, \theta_2)-Q(r, \theta_2)]. \end{aligned}$$

Accordingly, by (2.15),

(2.17) $$P(r, \theta_1)-P(r, \theta_2)<\pi \text{ for } \theta_1<\theta_2$$

Condition (2.17) is thus a necessary condition that $f(z)$ be close-to-convex; it can be expressed in other equivalent forms:

(2.18) $$\arg f'(re^{i\theta_1})-\arg f'(re^{i\theta_2})<\pi+(\theta_2-\theta_1)$$

for $\theta_1<\theta_2$, provided $\arg f'(z)=p(z)$ is chosen as above to be continuous for $|z|<1$. That is,

$$\int_{\theta_1}^{\theta_2} Re\left\{1+re^{i\theta}\frac{f''(re^{i\theta})}{f'(re^{i\theta})}\right\}d\theta>-\pi$$

for $\theta_1<\theta_2$.

(Sufficiency). Given $f(z)$ analytic for $|z|<1$, we choose $p(z)=\arg f'(z)$

to be continuous and then define $P(r, \theta)$ by (2.14). The condition (2.12) is then replaced by (2.17). In addition,

$$P(r, \theta+2\pi)-P(r, \theta)=2\pi, \tag{2.19}$$

since $p(z)$ has period 2π with respect to θ.

We now set $P(\rho, \theta), =t(\theta)$, for a fixed $\rho<1$, and apply the lemma 2, denoting the corresponding function $s(\theta)$ by $s(\rho, \theta)$. For $r<\rho$ we define

$$q_\rho(r, \theta)=\frac{1}{2\pi}\int_0^{2\pi}\frac{(\rho^2-r^2)[s(\rho, \alpha)-\alpha]}{\rho^2+r^2-2\rho r\operatorname{Cos}(\alpha-\theta)}d\alpha, \tag{2.20}$$

so that $q_\rho(r, \theta)$ is harmonic for $r<\rho$. Moreover, the function

$$Q_\rho(r, \theta)=q_\rho(r, \theta)+\theta \tag{2.21}$$

is monotone increasing in θ for each fixed $r<\rho$. For, if $\theta_1<\theta_2$,

$$Q_\rho(r, \theta_2)-Q_\rho(r, \theta_1)=\frac{1}{2\pi}\int_0^{2\pi}\frac{(\rho^2-r^2)[s(\rho, \alpha+\theta_2)-s(\rho, \alpha+\theta_1)]}{\rho^2+r^2-2\rho r\cos\alpha}d\alpha$$

and, since $s(\rho, \alpha)$ is non-decreasing in α, the right hand side is positive or 0. Hence $Q_\rho(r, \theta)$ is non-decreasing in θ, so that $\frac{\partial Q_\rho}{\partial\theta}\geq 0$.

Since this derivative is harmonic, the equality is rulled out, so that $Q_\rho(r, \theta)$ is strictly increasing.

We now choose an analytic function $h_\rho(z)$ whose imaginary part is $q_\rho(r, \theta)$ and such that $Re[h_\rho(0)]=0$. Then set

$$\phi_\rho(z)=\int_0^z e^{h_\rho(z)}dz, \tag{2.22}$$

so that

$$\phi_\rho(0)=0, \quad |\phi_\rho'(0)|=1. \tag{2.23}$$

The function $\phi_\rho(z)$ is then analytic for $|z|<\rho$. Moreover,

$$Re\left[1+z\frac{\phi_\rho''(z)}{\phi_\rho'(z)}\right]=\frac{\partial Q_\rho}{\partial\theta}>0, \quad |z|<\rho. \tag{2.24}$$

Hence $\phi_\rho(z)$ is a convex function for $|z|<\rho$. Furthermore, since

$$|P(\rho, \theta)-s(\rho, \theta)|\leq\frac{1}{2}\pi,$$

we conclude from (2.20) and the Poisson integral for $p(r, \theta)$ in terms of $P(\rho, \theta)$ that

$$|P(r, \theta)-Q_\rho(r, \theta)|<\frac{1}{2}\pi \quad \text{for } r<\rho.$$

Accordingly,

$$Re\left\{\frac{f'(z)}{\phi_\rho'(z)}\right\}>0 \qquad \text{for } |z|<\rho, \tag{2.25}$$

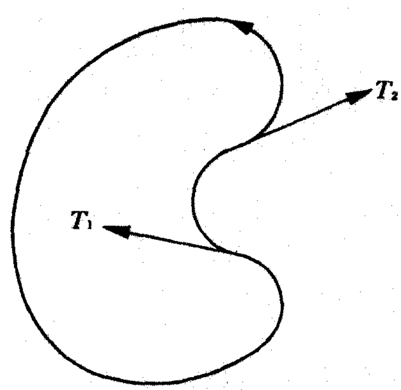
so that $f(z)$ is close-to-convex for $|z|<\rho$. It ramains to show that we can pass to the limit, $\rho \to 1$, and get a unique function $\phi(z)$ for $|z|<1$.

If we choose the sequence $\rho_n = 1 - 1/n$, then the corresponding functions $\phi_{\rho_n}(z)$ are defined for an increasing seqbence of domains. For each fixed n, the functions $\phi_{\rho_n}(z)$ for $m \geq n$ form a normal family for $|z|<\rho_n$; this follows from the normality of the family of normalized schlicht functions and condition (2.23). Hence a subsequence converges uniformly in this domain. By applying the diagonal process in the familiar fashion, we obtain a subsequence of $\phi_{\rho_n}(z)$ which converges uniformly in each circle $|z|<\rho<1$ and hence has as limit a unique function $\phi(z)$, analytic for $|z|<1$. Since the $\phi_\rho(z)$ are schlicht and convex, $\phi(z)$ must also be so, Since (2.25) holds for $\rho=\rho_n$, we conclude that

(2.26) $$Re\left[\frac{f'(z)}{\phi'(z)}\right]>0 \quad \text{for } |z|<1 \; ;$$

i. e., $f(z)$ is close-to-convex.

REMARK. (*Geometric interpretation for close-to-convex functions*). The condition (2.12) or its equivalent, condition (2.17), has the following geometric meaning: $w=f(z)$ maps each circle $z=re^{i\theta}$ (r fixed and $r<1$) onto a simple closed curve whose unit tangent vector $T=i\exp[ip(r, \theta)]$ either rotates in a counterclock-wise direction, as θ increases, or else rotates clockwise in such a manner that arg $T=p+\frac{1}{2}\pi$ never drops to a value π radians below a previous value; *i. e.*, $\triangle$ arg T exceeds $-\pi$, as θ increases. This is illustrated below. Here arg T_2 − arg T_1 is only slightly greater than $-\pi$.



Thus the geometric interpretation of (2.12) is that $w=f(z)$ maps each circle $|z|=r<1$ onto a simple closed curve whose tangent rotates, as θ increases, in such a way that it never turns back on itself sufficiently in the clockwise direction to reverse its direction completely.

III. SUBCLASSES OF CLOSE-TO-CONVEX FUNCTIONS

In this section we extend certain results due to Pommerenke [14], Robertson [22], Miller, Mocanu, and Reade [11].

DEFINITION. We shall say that $f(z)=\sum_{n=1}^{\infty} a_n z^n$, analytic for $|z|<1$, is of class K_α, $0\leq\alpha\leq 1$, if and only if $f(z)$ satisfies the following conditions:

(i) $f'(z)\neq 0$, for $|z|<1$

and (ii) the inequality

$$\int_{\theta_1}^{\theta_2} Re\left[1+z\frac{f''(z)}{f'(z)}\right]d\theta > -\pi\alpha, \qquad z=re^{i\theta}, \tag{3.1}$$

holds for all $\theta_1<\theta_2$ and for all r, $0\leq r<1$. We note that K_0 consists of convex functions, K_1 consists of close-to-convex functions.

For functions of class K_α, we have the following result:

THEOREM 3.1. If $f(z)=\sum_1^{\infty} a_n z^n\in K_\alpha$, then $f(z)$ is univalent and

$$|a_n|\leq[1+\alpha(n-1)]\,|a_1|, \qquad n=2,3,\cdots. \tag{3.2}$$

Proof) Since (3.2) reduces to a well-known result for $\alpha=0$, *i.e.*, for convex functions, we shall assume that $0<\alpha\leq 1$.

Kaplan [7] has shown that (3.1) implies the univalency of $f(z)$. Since (3.1) measures the change in the direction of a certain tangent vector, it follows that the function $f_1(z)=f(z)/a_1$ is again in K_α. An examination of the proof of Theorem 2.1 shows that there exists a convex univalent function $\phi(z)=\sum_1^{\infty} c_n z^n$ such that

$$\left|\arg\frac{f_1'(z)}{\phi'(z)}\right|<\frac{\pi\alpha}{2} \tag{3.3}$$

From (3.3) it follows that we may assume $c_1=e^{-i\beta}$, β real. Hence if we write

$$g(z)=\frac{f_1'(z)}{\phi'(z)}=\sum_0^{\infty} b_n z^n, \qquad b=e^{+i\beta}, \tag{3.4}$$

then we find

(3.5) $$(n+1)a_{n+1}=a_1\sum_{k=0}^{n}(k+1)c_{k+1}b_{n-k},\qquad n=1,2,3,\cdots.$$

For the function (3.4), subject to the condition (3.3), Littlewood [9] has shown that

(3.6) $$|b_n|\leq 2\alpha,\qquad n=1,2,3,\cdots$$

and for the convex function $\phi(z)$, with $c_1=e^{-i\theta}$, that

(3.7) $$|c_n|\leq 1,\qquad n=1,2,3,\cdots$$

hold, Hence, from (3.5), (3.6) and (3.7) we obtain

$$|a_n|\leqq[1+\alpha(n-1)]\,|a_1|,\qquad n=2,3,\cdots.$$

The question arises as to when equality holds in (3.2). It is clear that if equality holds for $n_0>1$, then equality must hold in (3.2) for all $1\leq n\leq n_0$. We can only exhibit a function for which equality holds for the case $n=2$. We merely choose

(3.8) $$g(z)=\left(\frac{1+z}{1-z}\right)^{\alpha}=1+2\alpha z+\cdots,$$

and $\phi(z)=z/(1-z)$ in (3.4); for the resulting $f(z)$ we find $a_1=1$, $a_2=1+\alpha$.

REMARK. For $\alpha=1$, (3.2) yields a verification of the Bieberbach conjecture for the close-to-convex functions [16].

DEFINTION. We introduce a class of functions K_α^* which bear much the same relation to star maps that the close-to-convex functions bear to convex maps. We shall say that $F(z)=\sum_{1}^{\infty}b_nz^n$, analytic for $|z|<1$, is in class K_α^*, $0\leq\alpha\leq 1$, if and only if $F(z)$ satisfies the following conditions:

(i) $b_1=F'(0)\neq 0$,

(ii) $F(z)=0$ if and only if $z=0$,

and (iii) the inequality

(3.9) $$\int_{\theta_1}^{\theta_2}Re\left(\frac{zF'(z)}{F(z)}\right)d\theta>-\pi\alpha,\qquad z=re^{i\theta},$$

holds for all $\theta_1<\theta_2$ and for all r, $0<r<1$. We see that K_0^* consists of all schlicht maps which are star with respect to $F(0)=0$, K_1^* consists of maps so called "close to-star".

For functions of class K_α^* we have the following results: We first note that, from the definitions of both the classes K_α and K_α^*, if $F(z)\in K_\alpha^*$ then

$f(z)=\int_{o}^{z} F(z)/z\ dz \in K_{\alpha}$, and if $f(z) \in K_{\alpha}$, then $F(z)=zf'(z) \in K_{\alpha}^{*}$.

THEOREM 3. 2. If $F(z)$ is analytic for $|z|<1$, if $F'(0) \neq 0$, and if $F(z)=0$ only when $z=0$, then a necessary and sufficient condition that $F(z) \in K_{\alpha}^{*}$, $0<\alpha \leq 1$, is that there exist a univalent map $\sigma(z)=\sum_{1}^{\infty} c_n z^n$, star with respect to $\sigma(0)=0$, such that

$$(3.10) \qquad \left| \arg \frac{F(z)}{\sigma(z)} \right| < \frac{\pi\alpha}{2}.$$

Proof) (Necessity) If $F(z) \in K_{\alpha}^{*}$, $0<\alpha \leq 1$ then it follows that $f(z)=\int_{o}^{z} F(z)/z\ dz \in K_{\alpha}$, and hence satisfies (3. 1). As in the proof of Theorem 3. 1, there exists a convex univalent function $\phi(z)$, $\phi(0)=0$ such that (3. 3) holds; that is

$$(3.11) \qquad \left| \arg \frac{f'(z)}{\phi'(z)} \right| = \left| \arg \frac{zf'(z)}{z\phi'(z)} \right| < \frac{\pi\alpha}{2}.$$

Now $\sigma(z)=z\,\phi'(z)$ is a star map of the form needed, so that (3.10) follows from (3.11).

(Sufficiency). If there is a star map satisfying (3.10), then

$$(3.12) \qquad \left| \arg \frac{F(re^{i\theta_2})}{\sigma(re^{i\theta_2})} - \arg \frac{F(re^{i\theta_1})}{\sigma(re^{i\theta_1})} \right| < \pi\alpha$$

for each pair of values $0 \leq \theta_1 \leq \theta_2 \leq 2\pi$. Since $\sigma(z)$ is a star map with respect to $\sigma(0)=0$, we know that $\arg \sigma(re^{i\theta_2}) - \arg \sigma(re^{i\theta_1}) > 0$ for $0 \leq \theta_1 \leq \theta_2 \leq 2$; hence (3.12) implies

$$(3.13) \qquad [\arg F(re^{i\theta_2}) - \arg F(re^{i\theta_1})] > -\pi\alpha$$

for all $0 \leq \theta_1 \leq \theta_2 \leq 2$ and for all r, $0 \leq r < 1$. But (3.13) is precisely (3. 9), so that $F(z) \in K_{\alpha}^{*}$, and the proof is now complete.

THEOREM 3. 3. If $F(z)=\sum_{1}^{\infty} b_n z^n \in K_{\alpha}^{*}$, then

$$(3.14) \qquad |b_n| \leq n[1+\alpha(n-1)]\,|b_1|, \quad n=2, 3, \cdots.$$

Proof) For $\alpha=0$, (3.14) is a known result for star maps. For $0<\alpha \leq 1$, (3.14) follows from Theorem 3. 1 and the relation $F(z)=zf'(z) \in K^{*}$ where $f(z) \in K_{\alpha}$. As in Theorem 3. 1, we can exhibit a function for which equality is achieved in (3.14) only for the case $n=2$.

DEFINITION. A function is called convex in one direction [19] if each line of a certain fixed direction intersects F in at most one interval.

We shall study another coefficient problem for the class K_α.

LEMMA 3. Let $0\leqq\alpha<1$, $f(z)=a_1z+a_2z^2+\cdots\in K_\alpha$. If $0\leqq r<1$ then

$$r\int_0^{2\pi}|f'(re^{i\theta})|\,d\theta\leqq J(\alpha)\max_{|z|=r} f(z) \tag{3.15}$$

where $J(\alpha)$ depends only on α.

The left side of (3.15) is equal to the length $L(r)$ of $\{f(z):|z|=r\}$. Since trivially $L(r)\geqq 2M(r)$, where $M(r)=\max_{|z|=r}|f(z)|$, inequality (3.15) shows that $L(r)$ and $M(r)$ have essentially the same growth.

Proof) Let $v(r,\theta)=arg\,[e^{i\theta}\phi'(re^{i\theta})]$. Then we have by (3.3) that

$$|arg\,[e^{i\theta}f'(re^{i\theta})]-v(r,\theta)|\leqq\frac{\alpha\pi}{2} \tag{3.16}$$

Let $m=[8/(1-\alpha)]+1$. For fixed $r<1$ we define θ_k by

$$v(r,\theta_k)=2\pi k/m \qquad (k=0,1,\cdots,m-1)$$

and $\theta_m=\theta_0+2\pi$. Since $v(r,\theta)$ increases with θ we see that

$$\theta_0<\theta_1<\cdots<\theta_m=\theta_0+2\pi$$

and that

$$0<v(r,\theta)-v(r,\theta_k)\leqq\frac{2\pi}{m}<(1-\alpha)\frac{\pi}{4}$$

for $\theta_k\leqq\theta\leqq\theta_{k+1}$. Together with (3.16) this gives us that

$$|arg\,[e^{i\theta}f'(re^{i\theta})]-v(r,\theta_k)|<(1+\alpha)\frac{\pi}{4}<\frac{\pi}{2}$$

for $\theta_k\leqq\theta\leqq\theta_{k+1}$. Hence

$$\cos\left((1+\alpha)\frac{\pi}{4}\right)\cdot r\int_{\theta_k}^{\theta_{k+1}}|f'(re^{i\theta})|\,d\theta$$

$$\leqq r\int_{\theta_k}^{\theta_{k+1}}|f'(re^{i\theta})|\cos\,[arg\,(e^{i\theta}f'(re^{i\theta}))-v(r,\theta_k)]\,d\theta$$

$$=Re\left[e^{-iv(r,\theta_k)}\int_{\theta_k}^{\theta_{k+1}}re^{i\theta}f'(re^{i\theta})\,d\theta\right]$$

$$=Re\,[e^{-iv(r,\theta_k)-i\pi/2}(f(re^{i\theta_{k+1}})-f(re^{i\theta_k}))]\leqq 2M(r)$$

for $k=0,1,\cdots,m-1$. By summation we obtain

$$r\int_{\theta_0}^{\theta_0+2\pi}|f'(re^{i\theta})|\,d\theta\leqq\frac{2mM(r)}{\cos[(1+\alpha)\pi/4]}\leqq\frac{18M(r)}{(1-\alpha)\sin[(1-\alpha)\pi/4]}$$

THEOREM 3. 4. Let $f(z)=\sum_{1}^{\infty}a_nz^n$ belong to K_α with $0\leqq\alpha<1$ or let $f(z)$ be convex in one direction. If

$$M(r)=\max_{|z|}|f(z)|\leqq\frac{B}{(1-r)^\lambda}\qquad(0\leqq r<1)$$

with $\lambda \geqq 0$, then

$$|a_n| \leq J\, Bn^{\lambda-1} \qquad (n=1, 2, \cdots)$$

where J depens only on α in the first case and $J \leqq 16/\pi$ in the second case.

Proof) (First case). Let $f(z) \in K_\alpha$ $(0 \leqq \alpha < 1)$. By Lemma 3, we have

$$(3.17) \qquad n\,|a_n|\,r^n = \left| \frac{r}{2\pi} \int_0^{2\pi} f'(re^{i\theta}) e^{-i(n-1)\theta} d\theta \right|$$

$$\leqq \frac{r}{2\pi} \int_0^{2\pi} |f'(re^{i\theta})|\, d\theta \leqq \frac{1}{2\pi} J(\alpha) M(r).$$

Taking $r = n/(n+1)$ we obtain $n\,|a_n| \leqq e\; J(\alpha)\; M(n/(n+1)) \leqq J_1(\alpha)\, Bn^\lambda$.

(Second case). Let $f(z) = a_1 z + \cdots$ be convex in the direction of the imaginary axis and

$$A(r) = \max_{|z|=r} |Re\ f(z)| \leqq B(1-r)^{-\lambda}$$

with $\lambda \geq 0$. Then we will show that

$$|a_n| \leqq \frac{16}{\pi} Bn^{\lambda-1}$$

If $f(z)$ is convex in the direction of the imaginary axis then for each fixed $r<1$ there are numbers θ_1 and θ_2 such that $u(r, \theta) = Re\ f(re^{i\theta})$ increases for $\theta_1 < \theta < \theta_2$ and decreases for $\theta_2 < \theta < \theta_1 + 2\pi$. Hence

$$(3.18) \qquad \int_0^{2\pi} \left| \frac{\partial}{\partial \theta} u(r, \theta) \right| d\theta \leqq 2 \left(\max_\theta u(r, \theta) - \min_\theta u(r, \theta) \right) \leqq 4A(r).$$

From the well-known representation [12] we obtain for $n = 1, 2, \cdots$

$$a_n = \frac{1}{\pi r^n} \int_0^{2\pi} u(r, \theta) e^{-in\theta} d\theta = \frac{1}{\pi i n r^n} \int_0^{2\pi} \left(\frac{\partial}{\partial \theta} u(r, \theta) \right) e^{-in\theta} d\theta.$$

Therefore (3.18) shows that

$$(3.19) \qquad |a_n| \leqq \frac{4}{\pi n r^n} A(r),$$

Let $A(r) \leqq B(1-r)^{-\lambda}$. With $r = (n-1)/n$ $(n = 2, 3, \cdots)$ we obtain from (3.19) that

$$|a_n| \leqq \frac{4}{\pi} B \left(\frac{n}{n-1} \right)^n n^{\lambda-1} \leqq \frac{16}{\pi} Bn^{\lambda-1}.$$

If $\lambda = 0$ we make $r \to 1-0$ in (3.19) and find $|a_n| \leqq (4/\pi)B$. This inequality is best possible as is shown by the functinon

$$\frac{2B}{\pi i} \log \frac{1+z}{1-z} = \frac{4B}{\pi i} \sum_{k=0}^\infty \frac{z^{2k+1}}{2k+1}$$

that is convex in the direction of the imaginary axis and satisfies $A(r) \leqq B$.

DEFINION. Let $f(z)=z+a_2z^2+\cdots$, be analytic for $|z|<1$ satisfying the conditions

(i) $f(z)f'(z)/z\neq 0$ for $|z|<1$

and (ii) for some nonnegative real number α and for some starlike function $\phi(z)=z+\cdots$,

$$(3.20)\qquad Re\left[(1-\alpha)\frac{zf'(z)}{\phi(z)}+\alpha\frac{(zf'(z))'}{\phi'(z)}\right]>0,\quad |z|<1.$$

For a fixed α, we denote this class of functions by M_α.

THEOREM 3. 5. Let $\alpha\geq 0$, and $\phi(z)$ be a starlike function in the open unit disk $|z|<1$ with $\phi(0)=0$, $\phi'(0)=1$. If $zf'(z)$ is analytic for $|z|<1$ with

$$zf'(z)\Big|_{z=0}=0,\qquad \left(zf'(z)\right)\Big|_{z=0}=1$$

and if

$$Re\left[(1-\alpha)\frac{zf'(z)}{\phi(z)}+\alpha\frac{(zf'(z))'}{\phi'(z)}\right]>0,\quad |z|<1$$

then

$$Re\left[\frac{zf'(z)}{\phi(z)}\right]>0,\quad |z|<1.$$

i. e., $f(z)$ is close-to-convex for $|z|<1$.

Proof) Let an analytic function $w(z)=z+b_2z^2+\cdots$be defined for $|z|<1$ by

$$(3.21)\qquad \frac{zf'(z)}{\phi(z)}=\frac{1-w(z)}{1+w(z)}$$

where $w(z)\neq -1$ for $|z|<1$. It suffices to shows that $|w(z)|<1$ for $|z|<1$ by subordinate principles [20]. Suppose $|w(z)|\geq 1$ for $|z|<1$, then by Jack's lemma [2] we could find η, $|\eta|<1$ such that $|w(\eta)|=1$ and

$$(3.22)\qquad \eta w'(\eta)=kw(\eta)\quad \text{where } k\geq 1.$$

We write

$$(3.23)\qquad \psi(z)=(1-\alpha)\frac{zf'(z)}{\phi(z)}+\alpha\frac{(zf'(z))'}{\phi'(z)}$$

From (3.21), (3.22) and (3.23), we have

$$(3.24)\qquad \psi(\eta)=\frac{1-w(\eta)}{1+w(\eta)}-\frac{2\alpha kw(\eta)}{(1+w(\eta))^2}\cdot\frac{\phi(\eta)}{\eta\phi'(\eta)}$$

with $|w(\eta)|=1$. Since $Re\dfrac{1-w(\eta)}{1+w(\eta)}=0$, $Re\dfrac{\phi(\eta)}{\eta\phi'(\eta)}>0$ and $\dfrac{w(\eta)}{(1+w(\eta))^2}$ is real and pouitive, therefore $Re\ \psi(\eta) \leqq 0$. This contradicts the given hypotheses that $Re\,[\psi(z)] > 0$ for $|z| < 1$. Then (3.21) gives us that

$$Re \frac{zf'(z)}{\phi(z)} > 0 \quad \text{for } |z| < 1.$$

Thus $f(z)$ is close-to-convex in $|z| < 1$.

THEOREM 3. 6. A function $f(z)$ is in the class M_α if and only if there exists a starlike function $\phi(z)$ and a function $p(z)$ which is regular and has positive real part for $|z| < 1$ such that

$$(3.25) \qquad f'(z) = \frac{1+c}{z[\phi(z)]^c} \int_0^z [\phi(\eta)]^c \phi'(\eta)\, p(\eta)\, d\eta$$

where $c = \frac{1}{\alpha} - 1$, $\alpha \neq 0$. If $\alpha = 0$, then $f'(z) = \left(\frac{\phi(z)}{z}\right) p(z)$.

Proof) Let $f(z) \in M_\alpha$ for $\alpha > 0$. If we set

$$(3.26) \qquad p(z) = (1-\alpha)\frac{zf'(z)}{\phi(z)} + \alpha \frac{(zf'(z))'}{\phi'(z)}$$

with $Re\ p(z) > 0$ for $|z| < 1$. Multiplying both sides of (3.26) by $\alpha^{-1}\,[\phi(z)]^c$ $\phi'(z)$ we obtain

$$(3.27) \qquad czf'(z)(\phi(z))^{c-1}\phi'(z) + (\phi(z))^c (zf'(z))' = (1+c)[\phi(z)]^c \phi'(z) P(z).$$

The left-hand side of (3.27) is the exact differential of $zf'(z)[\phi(z)]^c$, therefore, on integrating both sides of (3.27) with respect to z, we obtain (3.25). Conversely, if $f(z)$ satisfies (3.25), then it is easy to see that $f(z) \in M_\alpha$. On choosing $\phi(z) = \frac{z}{(1-z)^2}$ and $p(z) = \frac{1+z}{1-z}$ in (3.25), we obtain the following function of M_α,

$$(3.28) \qquad f(z) = \int_0^z (1+c)t^{-c-1}(1-t)^{2c}\left[\int_0^t \frac{\eta^c(1+\eta)^2}{(1-\eta)^{2c+4}}\, d\eta\right] dt,$$

where $c = \frac{1}{\alpha} - 1$.

IV. AN EXTENSION THE CONCEPTS OF CLOSE-TO-CONVEX FUNCTIONS TO MEROMORPHIC FUNCTIONS

In this section we extend the concepts of close-to-convex functions to meromorphic functions

$$(4.1) \qquad f(z) = \frac{1}{z} + a_0 + a_1 z + \cdots + a_n z^n + \cdots$$

regular in $0 < |z| < 1$ and with a simple pole at the origin.

DEFINITION. We say that $f(z)$, when given by (4.1), is close-to-convex in the punctured circle $0<|z|<1$ relative to $F(z)$ if there exists a meromorphic, starlike, schlicht function $F(z)$ in $|z|<1$ with a simple pole at the origin, given by

(4.2) $$F(z)=\frac{b_{-1}}{z}+b_0+b_1 z+\cdots+b_n z^n+\cdots (b_{-1}\neq 0),$$

such that

(4.3) $$Re\frac{zf'(z)}{F(z)}>0 \text{ for } |z|<1.$$

It is to be noticed that in the meromorphic case (4.3) does not imply that $f(z)$ is schlicht, as $Re\, zf'(z)/F(z)>0$, $(|z|<1)$ implies when $f(z)$ and $F(z)$ are regular in $|z|<1$. For example if $F(z)=-z^{-1}$ and

$$-z^2(1-z^2)f'(z)=1+z^2,\ f(z)=z^{-1}-2z-\frac{2}{3}z^3+\cdots (0<|z|<1),$$

then $f(z)$ is not schlicht in $0<|z|<1$, since $|a_1|>1$. Nevertheless, when $f(z)$ is meromorphic and satisfies (4.3), then $w=f(z)$ maps each circle $|z|=r$ $(0<r<1)$ onto a smooth curve which, although it may intersect itself, may also have "hairpin" turns, provided a complete reversal of the direction of the tangent does not occur. By a modification of Kaplan's argument [7], we obtain the following lemma :

LEMMA 4. Let $t(\theta)$ be a real function of θ for $-\infty<\theta<\infty$ such that

(4.4) $$t(\theta+2\pi)-t(\theta)=-2\pi,$$

(4.5) $$t(\theta_1)-t(\theta_2)>-\pi \quad \text{for } \theta_1<\theta_2 :$$

then there exists a real function $s(\theta)$ which is non-increasing and satisfies the conditions

(4.6) $$s(\theta+2\pi)-s(\theta)=-2\pi$$

(4.7) $$|s(\theta)-t(\theta)|\leq\pi/2.$$

Proof) The proof of the lemma 4 is the same as the proof of lemma 2, except that now $s(\theta)$ is defined as

(4.8) $$s(\theta)=\underset{\theta'\leq\theta}{g.l.b.}\, t(\theta')+\pi/2.$$

THEOREM 4.1. A necessary and sufficient condition that a meromorphic function $f(z)=\frac{1}{z}+a_0+a_1 z+\cdots$, regular in $0<|z|<1$ and with a simple pole at the origin, be close-to-convex in the punctured circle $0<|z|<1$ relative to $F(z)=\frac{b_{-1}}{z}+b_0+b_1 z+\cdots$, meromorphic and starlike in $|z|<1$, is that

$$(4.9)\qquad \int_{\theta_1}^{\theta_2} Re\left(1+re^{i\theta}\frac{f''(re^{i\theta})}{f'(re^{i\theta})}\right)d\theta<\pi$$

for $\theta_1<\theta_2$, $0<r<1$.

Proof) (Necessity). Let $f(z)$ and $F(z)$ be defined as in (4. 1) and (4. 2), and let (4. 3) hold. Using the notation of Theorem 2. 1, we let $i\exp iP(r,\theta)$ be the unit tangent vector to the image curve of $|z|=r$ $(0<r<1)$ through the mapping $w=f(z)$, and we let

$$Q(r,\ \theta)=\arg F(re^{i\theta}).$$

Because of (4. 3) (and with proper choice of the argument) we have

$$(4.10)\qquad |P(r,\ \theta)-Q(r,\theta)|<\pi/2.$$

Since $F(z)$ is meromorphic and starlike, we also have $\frac{\partial Q}{\partial\theta}<0$. Then, by an argument similar to that used in the proof of Theorem 2. 1, we obtain

$$(4.11)\qquad P(r,\ \theta_1)-P(r,\ \theta_2)>-\pi \text{ for } \theta_1<\theta_2,$$

and (4. 11) is readily seen to be equivalent to

$$(4.12)\qquad \int_{\theta_1}^{\theta_2} Re\left(1+re^{i\theta}\frac{f''(re^{i\theta})}{f'(re^{i\theta})}\right)d\theta<\pi.$$

Thus (4. 3) now implies (4. 12) instead of (2. 12).

(Sufficiency). If $f(z)$ is given by (4. 1) and $f'(z)\neq 0$ in $0<|z|<1$, we choose $p(z)=arg\ f'(z)$ to be continuous and then define $P(r,\ \theta)=p(re^{i\theta})+\theta$. The condition (4. 12) is then replaced by (4. 11). Since $f(z)$ is no longer analytic at $z=0$, we have

$$(4.13)\qquad P(r,\ \theta+2\pi)-P(r,\ \theta)=-2\pi.$$

Take $t(\theta)=P(\rho,\ \theta)$ $(0<\rho<1)$, and denote the corresponding $s(\theta)$ of the lemma 4 by $s(\rho,\ \theta)$. For $r>\rho$, define

$$(4.14)\qquad q_\rho(r,\ \theta)=\frac{\rho^2-r^2}{2\pi}\int_0^{2\pi}\frac{[s(\rho,\ \alpha)+\alpha]d\alpha}{\rho^2+r^2-2\rho r\cos(\theta-\alpha)},$$

so that $q_\rho(r,\ \theta)$ is harmonic for $r<\rho$. This definition of $q_\rho(r,\ \theta)$ differs from the one used by Kaplan [7] in the regular case, since $s(\rho,\ \alpha)-\alpha$ in his definition is replaced in (4. 14) by $s(\rho,\ \alpha)+\alpha$. Define the function

$$(4.15)\qquad Q_\rho(r,\ \theta)=q(r,\ \theta)-\theta$$

(replacing $q_\rho(r,\ \theta)+\theta$ used by Kaplan [7]). Then, since

$$s(\rho,\ \alpha+\theta_2)-s(\rho,\ \alpha+\theta_1)$$

has a period 2π, it follows by straightforward computation that for $\theta_1<\theta_2$

(4.16) $$Q_\rho(r,\ \theta_2)-Q_\rho(r,\ \theta_1)=\frac{\rho^2-r^2}{2\pi}\int_0^{2\pi}\frac{s(\rho,\ \alpha+\theta_2)-s(\rho,\ \alpha+\theta_1)}{\rho^2+r^2-2\rho r\cos\alpha}d\alpha<0.$$

We next define $h_\rho(z)$ to be the amalytic compltion of $q_\rho(r,\ \theta)$, so that the imaginary part of $h_\rho(z)$ is identical with the harmonic function $q_\rho(r,\ \theta)$, and such that $Re\ h_\rho(0)=0$. We define $F_\rho(z)$ by the equation

(4.17) $$F_\rho(z)=\frac{1}{z}\exp\ [h_\rho(z)]\neq 0,$$

and we write

(4.18) $$F_\rho(z)=\frac{c_{-1}}{z}+c_0+c_1z+\cdots \qquad (0<|z|<\rho).$$

Then

(4.19) $$c_{-1}=e^{h_\rho(0)},\quad |c_{-1}|=|e^{i\Im h_\rho(0)}|=1$$

$F_\rho(z)$ is analytic for $0<|z|<\rho$ and has a simple pole at the origin. We also have

(4.20) $$Re\frac{zF'_\rho(z)}{F_\rho(z)}=Re\,[zh'_\rho(z)-1]=\frac{\partial q_\rho(r,\ \theta)}{\partial\theta}-1=\frac{\partial}{\partial\theta}Q_\rho(r,\ \theta)<0.$$

Hence $F_\rho(z)$ is a starlike, schlicht function for $0<|z|<\rho$. Just like the argument of the proof of Theorem 2. 1, we choose a sequence $\rho_n\to 1$ so that $F_{\rho n}(z)\to F(z)$ uniformly in every closed domain within the unit circle. $F(z)$ is then schlicht and starlike in $0<|z|<1$, and

(4.21) $$Re\left(\frac{zf'(z)}{F(z)}\right)>0 \text{ for } 0<|z|<1.$$

Thus the proof is completed.

V. COEFFICIENT PROBLEM FOR CLOSE-TO-CONVEX FUNCTIONS

Let S denote the collection of regular and univalent functions

(5. 1) $$f(z)=z+\sum_{n=1}^{\infty}A_nz^n$$

in the unit disk $|z|<1$. Also, let S^* denote the subclass of S consisting of functions which map the unit disk onto domains which are starlike with respect to the origin. Let K denote the class of close-to-convex functions in S, *i. e.*, those functions $f(z)\in S$ such that there exists a $g(z)\in S^*$ and a real α such that $e^{i\alpha}zf'(z)/g(z)=p(z)$, where $p(z)$ is a regular function of positive real part in $|z|<1$.

M. S. Robertson [21] considered a generalized Bieberbach inequality

(5.2) $$||n|A_n|-m|A_m|| < |n^2-m^2||A_1|,$$

for m and n nonnegative integers. Robertson showed that (5.2) holds for all functions convex in one direction and that (5.2) holds for all functions in K if n-m is an even integer. He made no conjecture concerning its general validity for S. Indeed, it is not in general valid and in particular fails for $m=3$, and $n=2$. In this case the inequality (5.2) would be $|3|A_3|-2|A_2||<5$. But, *J. A.* Jenkins [6] showed that the sharp upper bound for $3|A_3|-2|A_2|$ is $(5.02\cdots)|A_1|$ when $f(z)\in S$. It was an open question whether (5.2) is true for all close-to-convex functions if $n-m$ is not an even integer. In 1968, Robertson stated that (5.2) would be true for all functions in K if every starlike function

$$g(z)=z+\sum_{n=2}^{\infty} b_n z^n, \quad |z|<1,$$

has coefficients with satisfy

(5.3) $$|2b_n-b_2b_{n-1}|\leq 2, \quad n=1, 2, 3, \cdots \ (b=0).$$

In 1975, *R. W.* Barnard found an example of a function in S^* where (5.3) does not hold. He considered the function

$$F_\alpha(z)=\left(\frac{1-z}{1+z}\right)^\alpha \frac{z}{(1-z)^2}=z+\sum_{n=2}^{\infty} b_n(\alpha)z^n, \quad 0\leq\alpha\leq 2.$$

where $b_2(\alpha)=2(1-\alpha)$, $b_3(\alpha)=3-4\alpha+2\alpha^2$, $b_4(\alpha)=4(3-5\alpha+3\alpha^2-\alpha^3)/3$, and $b_5(\alpha)=(15-28\alpha+22\alpha^2-8\alpha^3+2\alpha^4)/3$.

In this section, we prove a conjecture raised by *M. S.* Robertson [21], namely,

$$||n|a_n|-m|a_m|| < |n_2-m^2|$$

for all $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ that are close-to-convex using the inequalities due to *T. H.* MacGregor [5], Lebedev and Milin [5]. We skip the proofs of the following three lemmas and the proofs can be found in references.

LEMMA 5. Let P denote the class of functions

$$p(z)=1+\sum_{n=1}^{\infty} p_n z^n$$

analytic and of positive real part in $|z|<1$. Let $p\in P$, and let $\lambda_n\geq 0$. If $q(z)=\sum_{n=1}^{\infty} \lambda_n p_n z^n$ is analytic in $|z|<1$ and $Re\ q(z)\leq M$ for some positive M, then

(5.4) $$\sum_{n=1}^{\infty} \lambda_n |p_n|^2 \leq 2M.$$

COROLLARY TO THE LEMMA 5. For each $p \in P$ and for each finite sequence $\{\lambda_k\}_{k=1}^{n}$ with $\lambda_k \geqq 0$, there exists a η with $|\eta|=1$ such that

(5.5) $$\sum_{k=1}^{n} \lambda_k |p_k - \eta^k|^2 \leq \sum_{k=1}^{n} \lambda_k.$$

Proof of the Corollary. Apply the lemma 5 to $q(z)=\sum_{k=1}^{\infty} \lambda_k p_k z^k$. This gives

(5.6) $$\begin{aligned} \sum_{k=1}^{n} \lambda_k |p_k - \eta^k|^2 &= \sum_{k=1}^{n} \lambda_k |p_k|^2 - 2\,Re\,q(\bar{\eta}) + \sum_{k=1}^{n} \lambda_k \\ &\leqq 2M - 2\,Re\,q(\bar{\eta}) + \sum_{k=1}^{n} \lambda_k. \end{aligned}$$

Choosing η with $|\eta|=1$ so that $Re\, q(\bar{\eta}) = M = \max_{|z|=1} Re\{q(z)\}$, we obtain the result.

LEMMA 6. (Lebedev and Milin) [5]. If $\sum_{k=0}^{\infty} D_k z^k = \exp(\sum_{k=1}^{\infty} A_k z^k)$, and both functions are analytic in $|z|<1$, then

(5.7) $$\sum_{k=0}^{n-1} |D_k|^2 \leqq n\ exp\ \left\{ \sum_{k=1}^{n} \left(k\,|A_K|^2 - \frac{k^2\,|A_k|^2}{n} - \frac{1}{k} \right) + 1 \right\},$$

with equality if and only if $A_k = c^k/k$, $|c|=1$, $k=1, 2, \cdots; n-1$.

LEMMA 7. (D. Aharonov) [2]. With the notations in Lemma 6, we have

(5.8) $$\begin{aligned} &\sum_{k=0}^{n-1} |D_k|^2 + \frac{|D_n|^2}{2} \\ &\leqq \frac{2n+1}{2} exp \left\{ \sum_{k=1}^{n} \left(\frac{k^2}{n(2n+1)} + k - \frac{k^2}{n} \right) |A_k|^2 - \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{2n+1} + 1 \right\} \end{aligned}$$

with equality if and only if $A_k = c^k/k$, $|c|=1$, $k=1, 2, \cdots, n$.

LEMMA 8. If $g(z)=z+\sum_{n=2}^{\infty} b_n z^n$ is in S^*, for each positive integer n, there exists a η with $|\eta|=1$ such that

(5.9) $$1 + \sum_{k=1}^{n-1} |b_{k+1} - \eta b_k|^2 + \frac{|b_{n+1} - \eta b_n|^2}{2} \leqq n + \frac{1}{2}.$$

Proof) If $g(z) \in S^*$, then it is well known that

(5.10) $$\frac{zg'(z)}{g(z)} = p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n,$$

with $p(z)$ in P. Hence

(5.11) $$\log\frac{g(z)}{z}=\sum_{n=1}^{\infty}\frac{p_n}{n}z^n$$

Take a η on the unit circle, then

(5.12) $$(1-\eta z)\frac{g(z)}{z}=1+\sum_{k=1}^{\infty}(b_{k+1}-\eta b_k)z^k$$
$$=\sum_{k=0}^{\infty}D_k z^k,\ \text{say.}$$

On the other hand,

(5.13) $$\log(1-\eta z)\frac{g(z)}{z}=\sum_{k=1}^{\infty}\frac{(p_k-\eta^k)}{k}z^k$$
$$=\sum_{k=1}^{\infty}A_k z^k,\ \text{say.}$$

Then,

(5.14) $$\sum_{k=0}^{\infty}D_k z^k=\exp\left(\sum_{k=0}^{\infty}A_k z^k\right)$$

By Lemma 7, $\sum_{k=0}^{n-1}|D_k|^2+\frac{|D_n|^2}{2}$
$$\leq\frac{2n+1}{2}\exp\left\{\sum_{k=1}^{n}\left(\frac{k^2}{n(2n+1)}+k-\frac{k^2}{n}\right)|A_k|^2-\sum_{k=1}^{n}\frac{1}{k}-\frac{1}{2n+1}+1\right\}$$

(5.15)
$$=\frac{2n+1}{2}\exp\left\{\sum_{k=1}^{n}\left(\frac{k^2}{n(2n+1)}+k-\frac{k^2}{n}\right)\frac{|p_k-\eta^k|^2}{k^2}-\sum_{k=1}^{n}\frac{1}{k}-\frac{1}{2n+1}+1\right\}$$

By the corollary to Lemma 5, we can pick a η with $|\eta|=1$ such that

$$\sum_{k=1}^{n}\left(\frac{k^2}{n(2n+1)}+k-\frac{k^2}{n}\right)\frac{|p_k-\eta^k|^2}{k^2}$$
(5.16) $$\leq\sum_{k=1}^{n}\left(\frac{k^2}{n(2n+1)}+k-\frac{k^2}{n}\right)\frac{1}{k^2}$$
$$=\sum_{k=1}^{n}\left(\frac{1}{n(2n+1)}+\frac{1}{k}-\frac{1}{n}\right)$$
$$=\frac{1}{2n+1}+\sum_{k=1}^{n}\frac{1}{k}-1.$$

Hence, there exists a η with $|\eta|=1$ to make the exponent in (5.15) nonpositive, and we have

(5.17) $$\sum_{k=0}^{n-1}|D_k|^2+\frac{|D_n|^2}{2}\leq\frac{2n+1}{2}=n+\frac{1}{2}$$

This is the statement of Lemma 8.

THEOREM 5.1. If $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ is close-to-convex for $|z|<1$, then

(5.18) $$\left|\, n|a_n| - m|a_m| \,\right| < |\, n^2 - m^2 \,|$$

for all positive n and m. Strict inequality holds for all n and m, with $n \neq m$, unless $f(z)$ is a rotation of the Koebe function $z/(1-z)^2$.

Proof) If $f(z)\in K$ for $|z|<1$, then $e^{i\alpha} z f'(z)=r(z)g(z)$, where $Re\, r(z) > 0$ and

(5.19) $$r(z)=e^{i\alpha}+\sum_{k=1}^{\infty} r_k z^k.$$

For $|\eta|=1$, we have

(5.20) $$e^{i\alpha}(1-\eta z)f'(z)=(1-\eta z)\frac{g(z)}{z} r(z),$$

or

(5.21) $$e^{i\alpha}\left\{ 1+\sum_{n=1}^{\infty}\left[(n+1)a_{n+1}-\eta_n a_n\right] z^n \right\}$$
$$=\left[1+\sum_{k=1}^{\infty}(b_{k+1}-\eta b_k)z^k \right]\left\{ e^{i\alpha}+\sum_{k=1}^{\infty} r_k z^k \right\}$$

Comparing the coefficients of z^n on both sides, we get

(5.22) $$e^{i\alpha}\left[(n+1)a_{n+1}-\eta_n a_n\right]=r_n+r_{n-1}(b_2-\eta b_1)+\cdots+e^{i\alpha}(b_{n+1}-\eta b_n)$$
$$=\sum_{k=0}^{n} r_{n-k} D_k,$$

where $r_0=e^{i\alpha}$ and $D_k=b_{k+1}-\eta b_k$. To the first n terms of the right-hand side of (5.22), we apply the inequality $|ab|\leq |a/2|^2+|b|^2$, and to the last term the inequality $|b|\leq\frac{1}{2}+|b|^2/2$. We obtain,

$$|(n+1)a_{n+1}-\eta n a_n|$$

(5.23) $$\leq \left|\frac{r_n}{2}\right|^2+1+\left|\frac{r_{n-1}}{2}\right|^2+|b_2-\eta b_1|^2+\cdots+\frac{1}{2}+\frac{1}{2}|b_{n+1}-\eta b_n|^2$$
$$\leq n+\frac{1}{2}+1+\sum_{k=1}^{n-1}|b_{k+1}-\eta b_k|^2+\frac{1}{2}|b_{n+1}-\eta b_n|^2,$$

where we applied the well-known estimate $|r_k|\leq 2$, for $k=1, 2, \cdots$. By Lemma 8,

(5.24) $$|(n+1)a_{n+1}-\eta n a_n|\leq n+\frac{1}{2}+n+\frac{1}{2}=2n+1,$$

for some η with $|\eta|=1$. But by the triangle inequality,

(5.25) $$\left|(n+1)|a_{n+1}|-n|a_n|\right|\leq |(n+1)a_{n+1}-\eta n a_n|.$$

Hence,

(5.26) $$\left|(n+1)|a_{n+1}|-n|a_n|\right|\leq 2n+1.$$

The general case of inequality of (5.18) follows from this by induction.

In the case of equality, a close inspection of the proof shows that $r(z)$ has the form $(1+tz)/(1-tz)$, with $|t|=1$. Hence, we have

$$zf'(z)=[(1+tz)/(1-tz)]g(z).$$

Multiplying both sides by $(1-tz)$, we get

$$(5.27)\qquad (1-tz)\sum_{n=1}^{\infty} na_n z^n=(1+tz)\sum_{n=1}^{\infty} b_n z^n.$$

Comparing the coefficient of z^{n+1} on both sides, we obtain

$$(5.28)\qquad (n+1)a_{n+1}-tna_n=b_{n+1}+tb^n$$

Hence,

$$(5.29)\qquad 2n+1=|(n+1)|a_{n+1}|-n|a_n||\leqslant|b_{n+1}+tb_n|.$$

So our starlike function $g(z)$ must be a rotation of the Koebe function and has the form $z/(1-tz)^2$; hence $f(z)$ is $z/(1-tz)^2$ also.

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