HONAM MATHEMATICAL JOURNAL 1
Volume 4, Number 1, July. 1982

Infinitesimal normal variations of hypersurfaces of an odd-dimensional sphere

By Jong Ki Cho

§0. Introduction

Recently, K. Yano and U-H. Ki studied infinitesimal variations of hypersurfaces of a Sasakian manifold and proved the followings:

THEOREM A(I. 91). *Let M be a complete hypersurface of a $(2n+1)$-dimensional Sasakian manifold $(n>1)$ such that the function λ of the (f, g, u, v, λ)-structure induced on M does not vanish almost everywhere. If an infinitesimal normal variation $\bar{x}^h = x^h + \mu C^h \varepsilon\ (\mu \neq O)$ of M preserves f, then M is an even-dimensional sphere S^{2n}.*

THEOREM B (I. 91). *Under an infinitesimal normal variation $\bar{x}^h = x^h + \mu C^h \varepsilon\ (\mu \neq O)$ of a hypersurface M of a $(2n+1)$-dimensional Sasakian man*ifold *such that the function λ does not vanish almost everywhere, if the variation preserves f, then the variation is parallel or the hypersurface is totally geodesic.*

In the present paper, we improve the Theorem *A* and *B* without the condition with respect to λ and characterize the hypersurface of an odd-dimensional sphere. Our main results appear in § 3 (See Theorem 2 and Theorem 3).

§ 1. Preliminaries

Let S^{2n+1} be a $(2n+1)$-dimensional sphere with radius 1 covered by a system of coordinate neighborhoods $\{W; x^h\}$ in a Euclidean space E, where here and throughout this paper the indices h, i, j, k and 1 run over the range $\{\bar{1}, \bar{2}, \cdots, \overline{2n+1}\}$. It is well known that S^{2n+1} admits a canonical contact metric structure $(F_i{}^h, g_{ji}, F^h)$, which is induced from the natural Kaehlerian structure equipped on E([5]). Then the structure tensors of S^{2n+1} satisfy

(1. 1) $\begin{cases} F_j^h F_i^j = -\delta_i^h + F_i F^h, & g_{kh} F_j^k F_i^h = g_{ji} - F_j F_i, \\ F_j^h F_h = 0, \quad F_j F_i^j = 0, & F_j F^j = 1, \end{cases}$

where $F_j = g_{ji} F^i$ and

(1 , 2) $$\nabla_j F_i^h = -g_{ji} F^h + \delta_j^h F_i, \quad \nabla_j F^h = F_j^h,$$

∇_j denoting the operator of coveriant differentiation with respect to the metric g_{ji}.

Let M be a $2n$- dimensional Riemannian manifold isometrically immersed in S^{2n+1} by the immersion

$$i : M \rightarrow S^{2n+1}$$

and identify $i(M)$ with M itself. In terms of local coordinates (y^a) of M and (x^h) of S^{2n+1} the immersion i is locally expressed by $x^h = x^h(y^a)$, where here and in the sequel, the indices a, b, c, d, and e run over the range $\{1, 2, \cdots 2n\}$ If we put $B_b^h = \partial_b x^h$, $\partial_b = \partial / \partial y^b$, then B_b^h are $2n$ linearly independent vectors of S^{2n+1} tangent to M. Since the immersion is isometric, the first fundamental tensor g_{cb} of M is given by

(1 . 3) $$g_{cb} = B_c^j B_b^i g_{ji}$$

We represent the unit normal to M by C^h. Then the tranforms $F_i^h B_c^i$ of B_c^i by F_i^h can be expressed as linear combinations of B_c^h and C^h, that is,

(1 . 4) $$F_i^h B_c^i = f_c^a B_a^h - u_c C^h,$$

where f_c^a is a tensor field of type (1 , 1) and u_c a 1 — form on M. And the transform $F_i^h C^i$ of C^i by F_i^h, being orthogonal to C^h, can be written as

(1, 5) $$F_i^h C^i = u^a B_a^h,$$

where $u^a = u_b g^{ab}$ and $(g^{ab}) = (g_{ab})^{-1}$. Similarly we can put

(1 . 6) $$F^h = v^a B_a^h + \lambda C^h,$$

where v^a is a vector field and λ a function on M.

Applying F to the both sides of (1 . 4) ~ (1 . 6) respectively and using (1, 1), (1 . 3) and these equations, we find (*cf.* [1], [2], [4])

(1 . 7) $$\begin{cases} f_c^e f_e^a = -\delta_c^a + u_c u^a + v_c v^a, \\ g_{de} f_c^d f_b^e = g_{cb} - u_c u_b - v_c v_b, \\ u_e f_c^e = -\lambda v_c, \quad v_e f_c^e = \lambda u_c, \\ f_e^a u^e = \lambda v^a, \quad f_e^a v^e = -\lambda u^a, \\ u_e u^e = v_e v^e = 1 - \lambda^2, \quad u_e v^e = 0. \end{cases}$$

Thus, M admits the so-called (f, g, u, v, λ)-structure ([8], [11]).

If we put $f_{cb}=f_c^e g_{eb}$, then we easily see from (1.7) that $f_{cb}=-f_{bc}$.

Denotning by ∇_c the operator of the van der Waerden-Bortolotti covariant differentiation along the hypersurface M, we obtain the equations of Gauss and Weingarten;

$$(1.8) \qquad \nabla_c B_b^h=h_{cb}C^h, \quad \nabla_c C^h=-h_c^a B_a^h$$

respectively, where h_{cb} is the second fundamental tensor of M with respect to C^h and $h_c^a=h_{cb}g^{ab}$.

Differentiating (1.4)~(1.6) covariantly along M and taking account of (1.1), (1.2), (1.8) and these equations, we find (*cf.* [1], [3], [4])

$$(1.9) \qquad \nabla_c f_b^a=-g_{cb}v^a+\delta_c^a v_b+h_{cb}u^a-h_c^a u_b,$$

$$(1.10) \qquad \nabla_c u_b=\lambda g_{cb}+h_{ce}f_b^e,$$

$$(1.11) \qquad \nabla_c v_b=f_{cb}+\lambda h_{cb},$$

$$(1.12) \qquad \nabla_c \lambda=-u_c-h_{ce}v^e.$$

We note from (1.11) and $v_e v^e=1-\lambda^2$ that the function $1-\lambda^2$ is almost everywhere nonzero on M.

Since the ambient manifold is unit sphere, the equations of Gauss and Codazzi are given respectively by

$$(1.13) \qquad k_{dcb}^{\;\;a}=\delta_d^a g_{cb}-\delta_c^a g_{db}+h_d^a h_{cb}-h_c^a h_{db},$$

$$(1.14) \qquad \nabla_c h_{ba}-\nabla_b h_{ca}=0,$$

$k_{dcb}^{\;\;a}$ being the Riemann-Christoffel curvature tensor of M.

§2. Infinitesimal variations of a hypersurface of S^{2n+1}

We now consider an infinitesimal variation of the hypersurface M of S^{2n+1} given by

$$(2.1) \qquad \bar{x}^h=x^h+\xi^h(y)\,\varepsilon,$$

ξ^h being a vector field on S^{2n+1} and ε an infinitesimal.

If we put $\xi^h=\xi^a B_a^h+\mu C^h$, then

$$(2.2) \qquad \bar{x}^h=x^h+(\xi^a B_a^h+\mu C^h)\,\varepsilon,$$

where ξ^a and μ are a vector field and a scalar function on M respectively. We then have

$$(2.3) \qquad \bar{B}_b^h=B_b^h+(\partial_b\xi^h)\,\varepsilon,$$

where $\bar{B}_b^h=\partial_b\bar{x}^h$ are $2n$ linearly independent vectors tangent to the carried hypersurfaces at the varied point $(\bar{x}^h)$.

We displace $\bar{B}_b^h$ back parallelly from $(\bar{x}^h)$ to (x^h) and denote by Γ_{ji}^h the Christoffel symbols formed with g_{ji}, then we have

$$\widetilde{B}_b^h = B_b^h + (\nabla_b \xi^h)\,\varepsilon, \tag{2.4}$$

neglecting the terms of order higher than one with respect to ε, where

$$\nabla_b \xi^h = \partial_b \xi^h + \Gamma_{ji}^h B_b^j \xi^i.$$

In the sequel we always neglect terms of order higher than one with respect to ε. Putting $\delta B_b^h = \widetilde{B}_b^h - B_b^h$, we have

$$\delta B_b^h = (\nabla_b \xi^a - \mu h_b^a)\, B_a^h \varepsilon + (\nabla_b \mu + h_{be} \xi^e)\, C^h \varepsilon \tag{2.5}$$

because of (1.8), (2.2) and (2.3).

We denote by $\bar{C}^h$ the unit normal to the varied hypersurface. Then we have ([6], [7]).

$$\bar{C}^h = C^h - \Gamma_{ji}^h \xi^j C^i \varepsilon - (\nabla^a \mu + h_e^a \xi^e)\, B_a^h \varepsilon, \tag{2.6}$$

and for the variation $\delta C^h = \widetilde{C}^h - C^h$, where $\widetilde{C}^h$ is a vector obtained by transporting $\bar{C}^h$ parallelly back from $(\bar{x}^h)$ to (x^h), we have

$$\delta C^h = -(\nabla^a \mu + h_e^a \xi^e)\, B_a^h \varepsilon$$

where $\nabla^a = g^{ae} \nabla_e$.

It is well known that under an infinitesimal variation (2.1) of a hypersurface of S^{2n+1}, the variations of the first and second fundamental tensors are given respectively by ([6], [10])

$$\delta g_{cb} = ((\pounds g_{cb} - 2\mu h_{cb})\,\varepsilon, \tag{2.7}$$

$$\delta h_{cb} = \{\pounds h_{cb} + \nabla_c \nabla_b \mu + \mu (g_{cb} - h_{ce} h_b^e)\}\,\varepsilon, \tag{2.8}$$

where $\pounds$ denotes the operator of the Lie derivative with respect to ξ^a, that is, $\pounds g_{cb} = \nabla_c \xi_b + \nabla_b \xi_c$, $\xi_c = \xi^b g_{cb}$,

$$\pounds h_{cb} = \xi^e \nabla_e h_{cb} + (\nabla_b \xi^e) h_{ec} + (\nabla_c \xi^e) h_{be}.$$

From (2.7) we have

$$\delta g^{ba} = (\pounds g^{ba} + 2\mu h^{ba})\,\varepsilon. \tag{2.9}$$

Yano and Ki obtained the following variations of structure tensors on the hypersurface of S^{2n+1}.

Lemma 1 ([9]). Under an infinitesimal variation (2.1) of a hypersurface of S^{2n+1}, the variations of the structure tensors of the (f, g, u, v, λ)-structure induced on the hypersurface are given by

$$\delta f_b^a = \{\pounds f_b^a + \mu (f_b^e h_e^a - h_b^e f_e^a) - u_b (\nabla^a \mu) + (\nabla_b \mu) u^a\}\,\varepsilon, \tag{2.11}$$

(2.12) $$\delta u^a = \{ -\pounds u^a + (\nabla^e \mu) f_e^a - \mu h_e^a u^e + \mu v^a \} \varepsilon,$$

(2.13) $$\delta u_b = \{ -\pounds u_b + \mu h_{be} u^e - f_b^e \nabla_e \mu + \mu v_b \} \varepsilon,$$

(2.14) $$\delta v^a = (\pounds v^a + \lambda \nabla^a \mu - \nabla^a \lambda) \varepsilon,$$

(2.15) $$\delta \lambda = \{ \pounds \lambda - v^e \nabla_e \mu \} \varepsilon.$$

When $\xi^a = 0$, the infinitesimal variation (2. 1) is said to be ***normal*** ([6]). The variation (2. 1) is said to be ***parallel*** when the tangent space at a point (x^h) of the original hypersurface and that at the corresponding point $(\bar{x}^h)$ of the varied hypersurface are always parallel ([6]). From (2. 5) we see that the infinitesimal normal variation $\bar{x}^h = x^h + \mu C^h \varepsilon$ of the hypersurface is parallel if and only if $\mu = \text{const.}$ ([6]).

§ 3. Infinitesimal normal variations preserving the structure tensor

We suppose that the normal variation

(3. 1) $$\bar{x}^h = x^h + \mu C^h \varepsilon, \quad \mu \neq 0$$

preserves f_b^a. Then we have from (2.11)

(3. 2) $$\mu (h_{ce} f_b^e + h_{be} f_c^e) + (\nabla_b \mu) u_c - (\nabla_c \mu) u \;= 0$$

because of $\xi^a = 0$.

If we take the skew-symmetric part of (3. 2), then we find

(3. 3) $$(\nabla_b \mu) u_c \;\; (\nabla_c \mu) u_b = 0.$$

Thus (3. 2) reduces to

(3. 4) $$h_{ce} f_b^e + h_{be} f_c^e = 0.$$

because of $\mu \neq 0$.

Transvecting (3. 4) with $u^c u^b$ and $u^c v^b$ respectively and using (1. 7), we find

(3. 5) $$\lambda \beta = 0, \quad \lambda (\alpha - \gamma) = 0,$$

where we have put

(3. 6) $$(1-\lambda^2)\alpha = h_{cb} u^c u, \quad (1-\lambda^2)\beta = h_{cb} u^c v^b, \quad (1-\lambda^2)\gamma = h_{cb} v^c v^b.$$

Transvection (3. 4) with f_a^c gives

$$-h_{ba} + (h_{be} u^e) u_a + (h_{be} v^e) v_a + h_{ae} f_b^g f_a^e = 0.$$

from which, taking the skew-symmetric part,

$$(h_{be} u^e) u_a - (h_{ae} u^e) u_b + (h_{ba} v^e) v_a - (h_{ae} v^e) v_b = 0.$$

If we transvect this with u^a and v^a respectively and take account of (1. 7) and (3. 6), then we obtain

(3. 7) $$h_{be} u^e = \alpha u_b + \beta v_b,$$

(3. 8) $$h_{be} v^e = \beta u_b + \gamma v_b$$

because the function $1-\lambda^2$ does not vanish almost everywhere. Using (1.7), (1.10), (1.11), (1.12) and (1.14), we can prove from (3.7) and (3.8) that α, β and γ are constant on M (See [4]). Thus, by differentiating (3.8) covariantly along M and substituting (1.10) and (1.11), we find

$$(\nabla_c h_{be}) v^e + h_b^e (f_{ce} + \lambda h_{ce}) = \beta (h_{ce} f_b^e + \lambda g_{cb}) + \gamma (f_{cb} + \lambda h_{cb}),$$

from which, taking the skew-symmetric part and using (1.14),

(3.9) $$(\beta + 1) h_{ce} f_b^e + \gamma f_{cb} = 0.$$

If we transvect this with f_a^b and make use of (1.7), (3.7) and (3.8), then we get

(3.10) $$(\beta+1) h_{cb} = \gamma g_{cb} + (\alpha\beta + \alpha - \gamma) u_c u_b + \beta\gamma v_c v_b + \beta(\beta+1)(u_c v_b + u_b v_c).$$

Differentiating also (3.7) covariantly and substituting (1.10) and (1.11), we find

$$(\nabla_c h_{be}) u^e + h_b^e (\lambda g_{ce} + h_{ca} f_e^a) = \alpha (\lambda g_{cb} + h_{ce} f_b^e) + \beta (f_{cb} + \lambda h_{cb})$$

since α and β are constant, from which, taking the skew-symmetric part and using (1.14) and (3.4),

$$h_b^e h_{ea} f_c^a = \alpha h_{ce} f_b^e + \beta f_{cb}.$$

Transvecting this with f_a^b and making use of (1.7), (3.7) and (3.8), we find

(3.11) $$h_{ce} h_b^e = \alpha h_{cb} - \beta g_{cb} + \beta(\beta+1) u_c u_b + (\beta^2 + \gamma^2 - \alpha\gamma + \beta) v_c v_b + \beta\gamma + \beta\gamma (u_c v_b + u_b v_c).$$

Since β is constant on M, from (3.5) we may consider only two cases: (1) $\beta = O$ on M, (2) $\lambda = O$ on M. In the first case we easily see that $\alpha = \gamma$ on M (See [4]). Thus (3.10) reduces to

$$h_{cb} = \alpha g_{cb},$$

that is, M is totally umbilical and consequently, by completeness, M is a $2n$-dimensional sphere S^{2n}.

In the next place, we consider the case in which $\lambda = 0$. In this case it follows from (1.12) and (3.8) that $\beta = -1$, $\gamma = 0$. Thus (3.10) yields

(3.12) $$h_{ce} h_b^e = \alpha h_{cb} + g_{cb}.$$

Differentiating (3.12) covariantly and taking account of the fact that α is constant on M, we find

$$(\nabla_d h_{ce}) h_b^e + h_c^e \nabla_d h_b^e = \alpha \nabla_d h_{cb}.$$

from which, taking the skew-symmetric part with respect to the indices d and c and using (1.14),

$$h_c^e \nabla_a h_{be} - h_b^e \nabla_c h_{be} = 0$$

and consequently

$$h_{ce} \nabla_a h_b^e = h_{be} \nabla_a h_c^e .$$

Therefore we have

$$2 h_{ce} \nabla_a h_b = \alpha \nabla_a h_{cb}.$$

Transvecting this equation with h_a^c and using (3.12), then we find

$$2 \alpha h_{ae} \nabla_a h_b^e + 2 \nabla_a h_{ba} = \alpha h_{ae} \nabla_a h_b^e,$$

which implies

$$\nabla_a h_{cb} = 0$$

On the other hand, from (3.12) we can see that the second fundamental tensor h_c^a has two constant principal curvatures σ_1, σ_2 and the multiplicities of σ_1, σ_2 are odd (See [4]). Thus, if M is complete, then M is a product of two odd-dimensional spheres.

Developed above we conclude

Theorem 2. Under an infinitesimal normal variation $\bar{x}^h = x^h + \mu C^h \varepsilon$, $(\mu \neq 0)$ of a complete hypersurface M of an odd-dimensional unit sphere S^{2n+1} $(n > 1)$, if the variation preserves f of the (f, g, u, v, λ)-structure induced on M, then M is an even-dimensional sphere or a product of two odd-dimensional spheres.

Transvecting (3.3) with u^b and making use of (1.7), then we find

(3.13) $$\nabla_b \mu = A u_b,$$

where we have put $(1-\lambda^2) A = u^e \nabla_e \mu$ because the function $1-\lambda^2$ does not vanish almost everywhere.

Differentiating (3.13) covariantly and substituting (1.10), we find

$$\nabla_c \nabla_b = (\nabla_c A) u_b + A (\lambda g_{cb} + h_{ce} f_b^e),$$

from which, taking the skew-symmetric part and using (3.4),

(3.14) $$(\nabla_c A) u_b - (\nabla_b A) u_c + 2 A h_{ce} f_b^e = 0 .$$

If we transvect this with u^b and make use of (1.7) and (3.8), then we obtain

$$(1-\lambda^2) \nabla_c A - (u^e \nabla_e A) u_c + 2 A \lambda (\beta u_c + \gamma v_c) = 0 ,$$

or, use (3.5)

$$(1-\lambda^2) \nabla_c A = (u^e \nabla_e A) u_c - 2 A \gamma \lambda v_c,$$

Substituting this into (3.14), we get

$$A \{ h_{ce} f_b^e - (\lambda\gamma) / (1-\lambda^2) (v_c u_b - v_b u_c) \} = 0$$

because the function $1-\lambda^2$ is nonzero almost everywhere, from which, using (3. 5) and (3. 9), $A\gamma\{f_{cb}+\lambda/(1-\lambda^2)(v_c u_b-v_b u_c)\}=0$.

Transvecting this with f^{cb} and using (1. 7), we find $A\gamma=O$ if $n>1$. Thus (3.13) gives $\nabla_b(\mu\gamma)=0$ since γ is constant on M. Therefore we have $\mu\gamma=$ const. and consequently $\mu=$const. or $\gamma=0$ on M.

We now consider the case in which $\gamma=0$. In this case, (1.10) and (3.10) reduces respectively to

(3.15) $$(\beta+1)\nabla_c u_b=\lambda g_{cb},$$

(3.16) $$(\beta+1)\{h_{cb}-\alpha u_c u_b-\beta(u_c v_b+u_b v_c)\}=0$$

with the aid of (3. 5) and (3. 9).

Since α and β are constant, by differentiating (3.16) covariantly and using (1.11) and (3.15), we get

$$(\beta+1)\{\nabla_d h_{cb}-\lambda\alpha(g_{dc}u_b+u_c g_{db})+\beta\lambda g_{dc}v_b+\beta u_c(f_{db}+\lambda h_{db})+\beta\lambda g_{db}v_c+$$
$$\beta u_b(f_{dc}+\lambda h_{dc})\}=0,$$

from which, using (3. 5) with $\gamma=0$,

$$(\beta+1)\{\nabla_d h_{cb}-\beta(u_c f_{db}+u_b f_{dc})\}=0.$$

If we take the skew-symmetric part of this with respect to the indices d and c and making use of (1.14), then we have

$$\beta(\beta+1)(u_c f_{db}-u_d f_{cb}+2u_b f_{dc})=0.$$

Transvection $f^{dc}u^b$ yields $\beta(\beta+1)=0$ because the function $1-\lambda^2$ does not vanish almost everywhere.

On the other hand, we see from (3. 5) that $\alpha\gamma=0$. Thus it follows from (1. 12) and (3. 8) that $\alpha(\beta+1)=0$. If $\beta=0$, then we have $\alpha=0$. Consequently (3.10) reduces to $h_{cb}=0$, that is, the hypersurface is totally geodesic because of $\gamma=0$. Secondly if $\beta=-1$, then from the first relationship of (3. 5) we have $\lambda=0$.

Summing up, we have $\mu=$const. or $\lambda=0$ or $h_{cb}=0$ on M.

Thus we have

Theorem 3. Under an infinitesimal normal variation $\bar{x}^h=x^h+\mu C^h\varepsilon$, $(\mu\neq 0)$ of a hypersurface M of an odd-dimensional sphere S^{2n+1}, $(n>1)$, if the variation preserves f of the (f, g, u, v, λ)-structure induced on M, then the variation is parallel or the function λ vanishes identically or M is totally geodesic.

As in the proof of Theorem 2, if $\lambda=0$, then we see from (3.12) that the second fundamental tensor h_c^a is parallel.

Therefore we have

Corollary 4. Under the same assumptions as those stated in Theorem 3, the variation is parallel or the second fundamental tensor is parallel.

Chosun University

Bibliography

[1] Blair, D. E., G, D. Ludden and K. Yano, *Hypersurfaces of an odd-dimensional sphere,* J. Diff. Geo., 5 (1971), 479-486.

[2] Chung, C.-M. and U-H. Ki, *Converse problems of* $S^n \times S^n$, Honam Math. J., 3 (1981), 3-12.

[3] Ki, U-H., J.-K. Cho and S. B. Lee, *Hypersurfaces with quasi-integrable* (f, g, u, v, λ)*-structure,* to appear.

[4] Ki, U-H. and J. S. Pak, *Hypersurfaces with normal* (f, g, u, v, λ)*-structure of an odd-dimensional sphere,* to appear in Kodai Math. J.

[5] Sasaki, S., *Almost contact manifolds* I, II, *Lecture notes,* Tohoku Univ. (1965).

[6] Yano, K., *Infinitesimal variations of submanifolds,* Kodai Math. J., 1 (1978), 30-44.

[7] Yano, K., *Infinitesimal variations hypersurfaces of a Kaehlerian manifold,* J. of Math. Soc. of Japan, 29 (1977), 287-301.

[8] Yano, K. and U-H. Ki, *On quasi-normal* (f, g, u, λ)*-structure,* Kodai Math. Sem. Rep., 24 (1972), 106-120.

[9] Yano, K. and U-H. Ki, *Infinitesimal variations of hypersurfaces of a Sasakian manifold, Tensor,* N. S., 33 (1979), 1-10.

[10] Yano, K., U-H. Ki and J. S. Pak, *Infinitesimal variations the Ricci tensor of a submanifold,* Kōdai Math, Sem. Rep., 29 (1978), 271-284.

[11] Yano, K. and M Okumura, *On* (f, g, u, v, λ)*-structures,* Kōdai Math. Sem. Rep., 22 (1970), 401-423.