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ON THE THEORY OF OBSTRUCTION

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§1. Introduction

Recently, the obstruction theory has been extended to

(i) the cross section theory of fiber spaces ([3], [6]),

(ii) the foliation theory ([1]),

(iii) the theory of characteristic classes ([5]),

(iv) the extension theory of group rings ([4]).

The purpose of this note is to prove one property with respect to (i) above. That is, we shall prove that under some conditions for a fiber bundle

$\pi : E \to X$ (X is a finite CW-complex and each fiber is n-simple)

$$A^n_{s_o} = Q^n_\theta(X, A; B_n) = H^n(X, A; B_n),$$

where A is a subcomplex of X (Theorem13).

In §§ 2 and 3, we have explained and proved some terminologies and some properties with respect to the obstruction theory which is needed to understand Theorem 13. Finally, in § 4 we shall prove Theorem 13.

§2. Preliminaries

Let $\Delta^n = (e_o, \cdots e_n)$ be the standard n-simplex. For a CW-complex and a singular n-simplex $f : \Delta^n \to X$ with $f(e_o) = x_o$, there exists a continuous map $\widetilde{f} : \Delta^n \times \pi^{-1}(x_o) \to Y$ satisfying the conditions :

$$\pi_o \widetilde{f}(z, y) = z, \quad \widetilde{f}(e_o, y) = y,$$

where $\pi : Y \to X$ is a fiber bundle. For any two points x_o and x_1 we put

$\Omega(X; x_o, x_1) = \{ \omega : [0, 1] \to X \mid \omega$ is continuous with $\omega(0) = x_o$ and $\omega(1) = x_1 \}$.

Then, for each $\omega \in \Omega(X; x_o, x_1)$ there is a continuous map

$$\widetilde{\omega} : [0, 1] \times \pi^{-1}(x_o) \to Y \text{ such that}$$

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$$\pi\circ\widetilde{\omega}(t,y)=\omega(t), \qquad \widetilde{\omega}(0, y)=y.$$

Furthermore, if we put $\hat{\omega}=\widetilde{\omega}\,|\,1\times\pi^{-1}(x_0) : \pi^{-1}(x_o)\to\pi^{-1}(x_1)$ then the homotopy class of $\hat{\omega}$ is determind by the homotopy class α of ω. If we put

$[\pi^{-1}(x_o), \pi^{-1}(x_1)]$=the set of homotopy classes from $\pi^{-1}(x_o)$ to $\pi^{-1}(x_1)$,

we write the above fact by $\hat{\alpha}\in[\pi^{-1}(x_0), \pi^{-1}(x_1)]$. In particular, $\hat{\alpha}$ induces the isomorphism $\hat{\alpha} : H_n(\pi^{-1}(x_0);G)\to H_n(\pi^{-1}(x_1);G)$, where G is an abelian group.

In this note, we only consider the category F_b which is defined as follows. Every object of F_b is a fiber bundle $\pi : E\to X$ such that X is a finite CW-complex and each fiber is pathwise connected and n-simple ([2]).

Every morphism of F_b is a bundle map ([3]). Since each fiber of $\pi: E\to X$ in F_b is n-simple for each $x\in X$ and $\widetilde{x}\in\pi^{-1}(x)$ $\pi_n(\pi^{-1}(x), \widetilde{x})=\pi_n(\pi^{-1}(x))$ is abelian for all $n=1, 2, \cdots\cdots$. Moreover, for each $\omega\in\Omega(X;x_0,x_1)$ and $[\omega]=\alpha$ we have the isomorphism

$$\alpha^* : \pi_n(\pi^{-1}(x_o)) \longrightarrow \pi_n(\pi^{-1}(x_1)).$$

Let us put

$$B_n(x)=\pi_n(\pi^{-1}(x)),\ \ B_n(\alpha)=(\alpha^*)^{-1},\ \ (x\in X)$$

then we get a locally system over X ([2]).

$1_{\triangle^n}=1_n : \triangle^n\to\triangle^n$ gives the orientation of $\triangle^n$ by $[1_n]\in H^n(\triangle^n, \dot{\triangle}^n)$. Let $\sigma_\lambda : (\triangle^n, \dot{\triangle}^n)\to(X^{(n)}, X^{(n-1)})$ be a character map of a n-cell e_λ of X where $X^{(n)}$ is the n-dimensional skeleton. Then $\sigma_{\lambda *}([1_n])\in H_n(X^{(n)}, X^{(n-1)})=C_n(X)$ gives the orientation of e_λ. For a subcomplex A of X we put $\bar{X}^{(n)}=A\cup X^{(n)}$. Suppose there is a cross section $s:\bar{X}^{(n)}\to E$. The obstruction theory rises from that whether or not there exists an extension $s' : X^{(n+1)}\to E$.

Definition 1. Under the above situation the $(n+1)$-cochain

$$c^{n+1}(s)\in C^{n+1}(X, A;B_n)=H^{n+1}(\bar{X}^{(n+1)}, \bar{X}^{(n)};B_n)$$

is defined as follows. For a $(n+1)$-cell e_λ of X and a character map $\sigma=\sigma_\lambda$: $(\triangle^{n+1}, \dot{\triangle}^{n+1})\to(\bar{X}^{(n+1)}, \bar{X}^{(n)})$ we have the induced cross section

$$\sigma^*s : \dot{\triangle}^{n+1}\to\sigma^*E,$$

where $\sigma^*\pi : \sigma^*E\to\triangle^{n+1}$ is the induced bundle of $\sigma : \triangle^{n+1}\to\bar{X}^{(n+1)}\subset X$. Hence $\sigma^*E\subset\triangle^{n+1}\times E$, and for each $u\in\dot{\triangle}^{n+1}$

$$(\sigma^*s)(u)=(u,\ s\,\sigma(u))\in\triangle^{n+1}\times E.$$

Since $S^n\approx\dot{\triangle}^{n+1}$ (homeomorphic) we may assume that $[\sigma^*s]\in\pi_n(\sigma^*E, \sigma^*s(e_o))$. We put

$$[\sigma^* s] = c^{n+1}(s, \sigma)$$

For $1\dot{\triangle}^{n+1} = \hat{1}_n : \dot{\triangle}^{n+1} \to \dot{\triangle}^{n+1}$ we have that

$$c^{n+1}(s, \sigma) = (\sigma^* s) * ([\hat{I}_n]) \in \pi_n(\sigma^* E),\ \sigma^* s(e_o)).$$

In the fiber bundle $\sigma^* \pi : \sigma^* E \to \triangle^{n+1}$, since $\triangle^{n+1}$ is contractible we have

$$\pi_n((\sigma^*\pi)^{-1}(e_o),\ (\sigma^* s)(e_o)) \cong \pi_n(\sigma^* E,\ (\sigma^* s)(e_o))$$

in the homotopy exact sequence of $\sigma^* E$ ([3], [5], and [6]). In our category every fibre of a fiber bundle is homotopic, and thus

$$B_n(\sigma(e_o)) = \pi_n(\pi^{-1}(\sigma(e_o)) \cong \pi_n(\sigma^* E,\ (\sigma^* s)(e_o)).$$

That is, we put

$$\begin{array}{ccc} \pi_n(\sigma^*(E), (\sigma^* s)(e_o)) & \xrightarrow{\cong} & B_n(\sigma(e_o)) \\ \Psi & & \Psi \\ c^{n+1}(s, \sigma) & \rightsquigarrow & c^{n+1}(s)(e_\lambda), \end{array}$$

where $c^{n+1}(s) \in C^{n+1}(X, A; B_n)$. $c^{n+1}(s)$ is called an ***obstruction cocycle*** of a cross section $s : \bar{X}^{n} \to E$. (That $c^{n+1}(s)$ is a cocycle can be proved, see [1], [2]).

Definition 2. We consider a fiber bundle $\pi : E \to X$ in our category F_b. Let $s_o, s_1 : \bar{X}^{(n)} \to E$ be two cross sections such that

(i) $s_o \mid A = s_1 \mid A$

(ii) $\exists$ a homotopy $h : \bar{X}^{(n-1)} \times I \to E$ ($I = [0, 1]$) such that

$$h : s_o \mid \bar{X}^{(n-1)}) \simeq s_1 \mid \bar{X}^{(n-1)} \quad \text{rel } A$$

(for symbols see [3]). We are going to make a n cochain $d^n(s_o, s_1, h) \in C^n(X, A; B_n)$ as follows.

The fiber bundle $\pi \times 1 : E \times I \to X \times I$ has the same fiber of E and also $E \times I \in F_b$. Then we have

$$(X \times I)^{(n)} = X^{(n)} = X^{(n)} \times \dot{I} \cup X^{(n-1)} \times I$$

and hence $\overline{X \times I}^{(n)} = A \times I \cup (X \times I)^{(n)}$. A cross section $\bar{h} : \overline{X \times I}^{(n)} \to E$ is defined as follows:

$$\bar{h}(x, t) = \begin{cases} (h(x, t),\ t), & (x, t) \in \bar{X}^{(n-1)} \times I \\ (s_o(x),\ 0), & x \in \bar{X}^{(n)},\ t = 0 \\ (s_1(x),\ 1), & x \in \bar{X}^{(n)},\ t = 1 \end{cases}$$

In this case $B_n(E \times I) = \bar{B}_n(E \times I)$ is the induced locally system from the locally system $B_n(E)$ (Note : $B_n(x, t) = B_n(x)$). For each n-cell e_λ of X we define

$d^n(s_o, s_1, h)(e_\lambda) = (-1)^n c(\bar{h})(e_\lambda \times I) \in \bar{B}_n(\sigma_\lambda(e_o) \times 0) = B_n(\sigma_\lambda(e_o))$, and

$d^n(s_0, s_1, h)$ is called a *difference cochain* of s_0 and s_1. If $s_0 \mid \bar{X}^{(n-1)} = s_1 \mid \bar{X}^{(n-1)}$ and $h(x, t) = s_0(x)$ $((x,t) \in \bar{X}^{(n-1)} \times I)$ then we put

$$d(s_0, s_1, h) = d(s_0, s_1).$$

There are many properties with respect to obstruction cochains (proofs are omitted) ([2], [6]).

Property 3. Under the above circumstance

$$\delta d^n(s_0, s_1, h) = c^{n+1}(s_1) - c^{n+1}(s_0).$$

Property 4. For $n \geq 1$ if a cross section $s_0 ; \bar{X}^{(n)} \to E$ and a cochain $d \in C^n(X, A; B_n)$ then there exists a cross section $s_1 : \bar{X}^n \to E$ such that

$$s_0 \mid \bar{X}^{(n-1)} \text{ and } d^n(s_0, s_1) = d.$$

Property 5. For a fiber bundle $\pi : E \to X \in F_b$ let $s_0, s_1, s_3 : \bar{X}^{(n)} \to E$ be cross sections such that

$$h : s_0 \mid \bar{X}^{(n-1)} \simeq s_1 \mid \bar{X}^{(n-1)} \text{ rel } A, \quad h' : s_1 \mid \bar{X}^{(n-1)} \simeq s_2 \mid \bar{X}^{(n-2)} \text{ rel A}.$$

Then for a homotopy $k : \bar{X}^{n-1)} \to E$ defined as

$$k(x, t) = \begin{cases} h(x, 2t), & 0 \leq t \leq \frac{1}{2} \\ h'(x, 2t-1), & \frac{1}{2} \leq t \leq 1 \end{cases}$$

we have

$$d^n(s_0, s_1, h) + d^n(s_1, s_2, h') = d^n(s_0, s_2, k).$$

§ 3. Obstruction sets and $d^n(s_0, s_1, h)$

Definition 6. For a fiber bundle $\pi : E \to X$ and a subcomplex A of X, let $s : A \to E$ be a cross section. The *$(n+1)$-dimensional obstruction set*

$$O^{n+1}(s) \in H^{n+1}(X, A; B_n)$$

is defined as follows. If s is not extended to $\bar{X}^{n)} \to E$ then $O(s)$ is the vacuous set. Suppose s is extended to $s' : \bar{X}^{(n)} \to E$ then we have the cohomology class $[c^{n+1}(s')] \in H^{n+1}(X, A; B_n)$ which is called the *$(n+1)$-dimensional obstruction element* of s. We define

$$O^{n+1}(s) = \{ (n+1)\text{-dimensional obstruction elements of } s \}.$$

By our defintion (Definition 6) the following can be easily proved.

(i) As section $s_0 \simeq s_1 : A \to E$ then $O^{n+1}(s_0) = O^{n+1}(s_1)$,

(ii) If a cross section $s : A \to E$ can be extended to $\bar{X}^{n+1)} \to E$ then $O^{n+1}(s)$

contains the zero element of $H^{n+1}(X, A; B_n)$.

(iii) if for every $n \geq 1$

$$H^{n+1}(X, A; B_n) = 0$$

then every cross section $s : A \to E$ can be extended to a total cross section $X \to E$.

Let us consider two cross sections $s_o, s_1 : X \to E$ such that $s_o \mid A = s_1 \mid A$. If there is a homotopy

$$h : \bar{X}^{(n-1)} \times I \to E \text{ such that}$$
$$h : s_o \mid \bar{X}^{(n-1)} \simeq s_1 \mid \bar{X}^{(n-1)} \text{ rel } A$$

then we have a difference cochain $d^n(s_o, s_1, h) \in C^n(X, A; B_n)$.

Lemma 7. *Under the above situation $d^n(s_o, s_1, h)$ is a cocycle.*

Proof . By property 3

$$\delta d^n(s_o, s_1, h) = c^{n+1}(s_1) - c^{n+1}(s_o).$$

Since for a cross section $s : \bar{X}^{(n)} \to E$

s is extended to $\bar{X}^{(n+1)} \to E \Leftrightarrow c^{n+1}(s) = 0$, $c^{n+1}(s_1) = c^{n+1}(s_o) = 0$. Therefore $\delta d^n(s_o, s_1, h) = 0$. ▮

Lemma 8. *Under the above situation the homotopy h has an extension h^* : $\bar{X}^{(n)} \to E$ such that*

$$h^* : s_o \mid \bar{X}^{(n)} \simeq s_1 \mid \bar{X}^{(n)} \text{ rel } A,$$

if and only if $d^n(s_o, s_1, h) = 0$.

Proof. Let us put

$$J = X \times I \qquad M = A \times I \cup X \times 0 \cup X \times 1$$
$$\bar{J}^{(n)} = M \cup J^{(n)} = (X \times 0) \cup (\bar{X}^{(n-1)} \times I) \cup (X \times 1).$$

Define a map $F : \bar{J}^{(n)} \to E$ by taking

$$F(x, t) = \begin{cases} s_o(x), & x \in X, \ t = 0, \\ h(x, t), & x \in \bar{X}^{(n-1)}, \ t \in I, \\ s_1(x), & x \in X, \ t = 1. \end{cases}$$

Then F determines an obstruction cocycle $c^{n+1}(F)$ of the CW-complex J modulo M. Since there is the isomorphism

$$\begin{array}{ccc} \psi : C^n(X, A; B_n) & \longrightarrow & C^{n+1}(J, M; B_n) \\ \Cup & & \Cup \end{array}$$

$d^n(s_o \mid \bar{X}^{(n)}, s_1 \mid \bar{X}^{(n)}, h) \mapsto (-1)^{n+1}(c^{n+1}(F) - c^{n+1}(s_o \mid \bar{X}^{(n)}) \times 0 - c^{n+1}(s_1 \mid \bar{X}^{(n)}) \times 1)$ ([2]), we have

$$\psi d^{n}(s_{o}\mid \bar{X}^{(n)},\ s_{1}\mid \bar{X}^{(n)},\ h)=(-1)^{n+1}\ c^{n+1}(F).$$

Since F has an extension $F':\bar{J}^{(n+1)}\to E\Leftrightarrow c^{n+1}(F)=0$ and ψ is the isomomorphism

$$\exists\ h^{*}\Leftrightarrow c^{n+1}(F)=0\Leftrightarrow d^{n}(s_{o}\mid \bar{X}^{(n)},\ s_{1}\mid \bar{X}^{(n)},\ h)=0. \blacksquare$$

By Lemma 7 the cocycle $d^{n}(s_{o}\mid \bar{X}^{(n)}, s_{1}\mid \bar{X}^{(n)}, h)$ represents an ***obstruction cohomology class***

$$[d^{n}(s_{o}\mid \bar{X}^{(n)}, s_{1}\mid \bar{X}^{(n)}, h)]=\delta^{n}(s_{o}, s_{1}, h)\in H^{n}(X, A; B_{n}).$$

Definition 9. In our category F_b for a fiber bundle $\pi: E\to X$, let $s: X\to E$ be a cross section. We set

$$\Omega=\{\ \omega:\bar{X}^{(n+1)}\to E\mid s \text{ is a cross sections}\ \}$$

$$W=\{\ \omega\in\Omega\ \big|\ s\mid A=\omega\mid A\ \}$$

and $\omega_{o}\in W$ such that $\omega_{o}=s\mid \bar{X}^{(n-1)}$. We define

$$R^{n}(X,\ A;\ s)=\pi_{1}(W,\omega_{o})$$

which is a group.

The each element α of $R^{n}(X,\ A;\ s)$ is represented by a homotopy

$$h:\bar{X}^{(n-1)}\times I\to E \text{ such that}$$

$$h: s\mid \bar{X}^{(n-1)}=s\mid \bar{X}^{(n-1)} \text{ rel } A.$$

Therefore we obtain an obstruction cohomology class $\delta^{n}(s_{o}, s_{o}, h)\in H^{n}(X,\ A;\ B_{n})$ which depends only on α. There is a homomorphism

$$\begin{array}{ccc} \xi_{n}: R^{n}(X,\ A;\ S) & \longrightarrow & H^{n}(X,\ A;\ B_{n}) \\ \Cup & & \Cup \\ \alpha & \rightsquigarrow & \delta^{n}(s_{o}, s_{o}, h). \end{array}$$

We also put $\xi_{n}(R^{n}(X,\ A;\ s))=J^{n}_{s}=J^{n}_{s}(X,\ A;\ B_{n})$ which is a subgroup of $H^{n}(X,\ A;\ B_{n})$.

We shall denote the quotient group

by $$Q^{n}_{s}=Q^{n}_{s}(X,\ A;\ B_{n})=H^{n}(X,\ A;B_{n})/J^{n}_{s}.$$

Definition 10. For a fiber bundle $\pi: E\to X\in F_b$ and two cross sections $s_{o}, s_{1}: X\to E$ with $s_{o}\mid A=s_{1}\mid A$ we shall define the n-dimensional obstruction set

$$O^{n}(s_{o}, s_{1})\subset H^{n}(X,\ A;\ B_{n})$$

as follows, where A is a subcomplex of X. If

$$s_{o}\mid \bar{X}^{(n-1)}\not\simeq s_{1}\mid \bar{X}^{(n-1)} \text{ rel } A$$

then $O^{n}(s_{o}, s_{1})=\phi$. Now suppose that there exists a homotopy $h:\bar{X}^{(n-1)}\times I\to E$ such that $h: s_{o}\mid \bar{X}^{(n-1)}\simeq s_{1}\mid X^{(n-1)}$ rel A.

Then there is the cohomology class $\delta^n(s_o, s_1, h) \in H^n(X, A; B_n)$ which is called an ***n-dimensional obstruction elements of*** s_o and s_1.

We defeine

$O^n(s_o, s_1) = \{ n\text{-dimensional obstruction element of } s_o, \text{ and } s_1 \}$.

In the above situation the following are easy to prove ([2]).

(i) For two cross sections $s_o, s_1 : X \to E$ with $s_o \mid A = s_1 \mid A$

$$s_o \mid \bar{X}^{(n)} \simeq s_1 \mid \bar{X}^{(n)} \text{ rel } A \Leftrightarrow O^n(s_o, s_1) = J^n_{s_o}(X, A; B_n),$$

(ii) s_o and s_1 are the same as above. If

$$s_o \mid \bar{X}^{(n-1)} \simeq s_1 \mid \bar{X}^{(n-1)} \quad \text{rel } A$$

then s_o and s_1 determine a unique element $\chi^n(s_o, s_1)$ of $Q^n_{s_o}(X, A; B_n)$. Moreover

$$s_o \mid \bar{X}^{(n)} \simeq s_1 \mid \bar{X}^{(n)} \text{ rel } A \Longleftrightarrow \chi^n(s_o, s_1) = 0.$$

For a fiber bundle $\pi : E \to X \in F_b$ and a fixed cross section $s : A \to E$, where A is a subcomplex of X. We put

$$S_s = \{ \omega : X \to E \mid \omega \text{ is a cross section with } s \mid A = \omega \mid A \}$$

Let θ be a given $(n-1)$-homotopy class relative to A of the maps S_s, *i. e.*, for $s_o, s_1 \in \theta$

$$s_o \mid \bar{X}^{(n-1)} \simeq s_1 \mid \bar{X}^{(n-1)} \text{ rel } A, \quad s_o \mid A = s_1 \mid A = s \mid A.$$

Then θ determines a subgroup $J^n_\theta(X, A; B_n)$ of $H^n(X, A; B_n)$ and hence the quotient group

$$Q^n_\theta(X, A; B_n) = H^n(X, A; B_n) / J^n_\theta(X, A; B_n)$$

(see Definition 9). We have to note that

$$s_o \mid \bar{X}^{(n-1)} \simeq s_1 \mid \bar{X}^{(n-1)} \text{ and } s_o \mid A = s_1 \mid A \Rightarrow J^n_{s_o} = J^n_{s_1}.$$

For each $s_o \in \theta$, by (ii) above every $s_1 \in \theta$ determines a characteristic element $\chi^n(s_o, s_1) \in Q^n_\theta(X, A; B_n)$. An element $\alpha \in Q^n_\theta(X, A; B_n)$ is said to be s_o-*admissible* if there is a cross section $s_1 \in \theta$ such that $\chi^n(s_o, s_1) = \alpha$. We put

$A^n_{s_o}$ = the set of all s_o-admissible elements in $Q^n_\theta(X, A; B_n)$, which is called the s_o-admissible set in $Q^n_\theta(X, A; B_n)$. By property 5, for each pair $s_o, s_1 \in \theta$ it is clear that

$$A^n_{s_o} = \chi^n(s_o, s_1) + A^n_{s_1}.$$

Lemma 11. *The n-homotopy classes relative to A of* S_s *which are contained in* θ *are in one-to-one correspondece with the elements of* $A^n_{s_o}$ *for an arbitrary element* s_o *of* θ.

Proof. For $s_1 \in \theta$ $\chi^n(s_o, s_1) \in A^n_{s_o}$ as before, where $s_o \in \theta$. $\chi^n(s_o, s_1)$ depends only on the n-homotopy class relative to A which contains s_1 because of that if $s_1 | \overline{X}^{(n)} \simeq s_2 | \overline{X}^{(n)}$ rel A and $s_1, s_2 \in \theta$ by (ii) above.

$$\chi^n(s_o, s_1) - \chi^n(s_o, s_2) = \chi^n(s_1, s_2) = 0.$$

Therefore, $\tau: s_1 \rightsquigarrow \chi^n(s_o, s_1)$ defines a correspondence from the n-homotopy classes relative to A contained in θ to $A^n_{s_o}$. That τ is onto follows from the defnition of $A^n_{s_o}$. Suppose $\chi^n(s_o, s_1) = \chi^n(s_o, s_2)$ then

$$\chi^n(s_1, s_2) = \chi^n(s_o, s_1) - \chi^n(s_o, s_2) = 0$$

which means that $s_1 | \overline{X}^{(n)} \simeq s_2 | \overline{X}^{(n)}$ rel A by (ii) above.

§ 4. Main Theorems

In this section, we shall consider a fiber bundle $\pi: E \to X \in F_b$ and some properties of cross section $s: X \to E$

Lemma 12. *For a given cross section $s_o: X \to E$, if dim $X \leq n+1$ then for each cohomology class $u \in Q^n_\theta(X, A; B_n)$ there exists a cross section $s_1: X \to E$ such that*

$$\chi(s_o, s_1) = u, \quad s_o | A = s_1 | A,$$

where A is a subcomplex of X.

Proof. Take $d \in C^n(X, A; B_n)$ with $[d] = u$. Then, by property 4 there exists an extension $s_1: \overline{X}^{(n)} \to E$ of $s_o | \overline{X}^{(n-1)}: \overline{X}^{(n-1)}) \to E$ such that $d^n(s_o | \overline{X}^{(n)}, s_1) = d$. By property 3. we have

$$c^{n+1}(s_1) - c^{n+1}(s_o) = \delta d(s_o | \overline{X}^{(n)}, s_1) = \delta d = 0.$$

Since $s_o: X \to E$ is a cross section $c^{n+1}(s_o) = 0$, and thus $c^{n+1}(s_1) = 0$.

This means that s_1 has an extension $\overline{X}^{(n+1)} = X \to E$, *i.e.*, s_1 is a cross section of the fiber bundle $\pi: E \to X$. ∎

Theorem 13. *Using notations an in* § 3, *we have*

$$A^n_{s_o} = Q^n_\theta(X, A; B_n) = H^n(X, A; B_n)$$

if dim $X \leq n+1$ and $s_o \in \theta$.

Proof. By Lemma 12, we have for a fixed element $s_o \in \theta$

$$\{\chi^n(s_o, s_1) \mid s_o, s_1 \in \theta\} = Q^n_\theta(X, A; B_n) = H^n(X, A; B_n) / J^n_\theta(X, A; B_n).$$

Since for $s_o, s_1 \in \theta$

$$J^n_{s_0}(X,\ A;\ B_n)=J^n_{s_1}(X,\ A;\ B_n)$$

we have

$$J^n_{s}(X,\ A;\ B_n)=J^n_{s_0}(X,\ A;\ B_n).$$

As in the proof of Lemma 11, since $\chi^n(s_0, s_1)$ depends only on the n-homotopy class relative to A which contains s_1 we have

$$J^n_{s_0}(X,\ A;\ B_n)=\{\chi^n(s_0, s_0)\}.$$

By (ii) of § 3 $\chi^n(s_0, s_0)=0$ because of that $s_0:\bar{X}^{(n+1)}=X\to E$ is a cross section. Therefore, by the definion of $A^n_{s_0}$

$$A^n_{s_0}=Q^n_{\theta}(X,\ A;\ B_n)=H^n(X,\ A;\ B_n).$$

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