HONAM MATHEMATICAL JOURNAL
Volume 4, Number 1, July. 1982

A Study on hereditary Noetherian prime rings

By Saki Min

1

The object of this paper is to give the various results on hereditary Noetherian prime rings. Most of these are well-known. an exeption being Theorem 3, which says that such a ring satisfies a restricted minimum condition.

2

We give, at first, some basic definitions and sketch some rather technical properties of an order R in a simple Artinian quotient ring Q.

A ***quotient ring*** Q is a ring, with 1, in which every ***regular*** element (i. e. not a zero divisor) has an inverse. A ***right order*** R in Q is a subring of Q such that every element Q has the form xy^{-1} for some elements x, y in R. Similary, we define a left (two-sided) order. A ring is ***prime*** if (0) is a prime ideal. By virture of Goldie's theorem ([2]) any Noetherian prime ring is a right and left order in a simple Artinian quotient ring.

A right ideal U of R is ***uniform*** if any nonzero submodules of U have a nonzero intersection : i. e. U contains no direct sum of right ideals. It is easy to show that if U is a uniform right ideal of R then UQ is a minimal right ideal of Q. Also, if M is a minimal right ideal of Q, then $M \cap R$ is a maximal uniform right ideal of R. Since M is the right annhilator in Q of an element of Q, it follows that any maximal uniform right ideal is the annhilator in R of some element of R.

A right ideal of a ring is said to be ***essential*** if it has nonzero intersection with each nonzero right ideal of the ring.

An example of Q is the $n \times n$ matrix ring over a division ring for some n, and any direct sum of n minimal right ideals inside Q is equal to Q. Thus we see that a right ideal of R is an essential submodule of R iff it contains a direct sum of n uniform right ideals ([2]). In general, if the direct sum of K uniform right ideals is essential in a right ideal I, we say that I has ***uniform dimension*** K. Any uniform right ideal contains a copy of any order. For, if U and V are uniform right ideals, then $UV \neq 0$ since R is prime. But VQ is a minimal right ideal of Q so that, for any $q \in Q$, $qVQ = O$ or $qx \neq O$ for each nonzero $x \in VQ$. Since there is an element $u \in U$ such that $uV \neq O$ We see that $U \supseteq uV \cong V$. One consequence of this is that, if n is the uniform dimension of R. then an arbitrary direct sum of n uniform right ideals is isomorphic to an essential right ideal.

3

Proposition 1. A Noetherian prime ring which contains a minimal right ideal is simple Artinian.

Proof. The minimal right ideal is certainly uniform. Since an isomorphic copy of every other uniform right ideal is contained in it, every uniform right ideal is minimal. Now there is a finite direct sum of uniform right ideals which is an essential right ideal and so contains a regular element, whence it contains a right ideal isomorphic to R. Thus R is a finite direct sum of simple right modules, all isomorphic. Hence R is a simple Artinian ring. QED.

Let R again be an order in a simple Artinian Q. A right R-submodule I of Q is called a *fractional right R-ideal* if there is a unit $q \in Q$ such that qI is an essential right ideal of R. It is clear that any homomorphism from I to another fractional right ideal can be extended to an endomorphism of Q and hence may be regarded as left multiplication by some element of Q. since I contains a unit of Q (qI contains a regular element of R) this extention is unique. Thus

$$I^* = \{ q \in Q \mid qI \subseteq R \} \cong Hom_R(I, R)$$

$$O_l(I) = \{ q \in Q \mid qI \subseteq I \} \cong Hom_R(I, I)$$

I^* is clearly a fractional left R-ideal. We will have occasion to use the following easy generlization of [1] proposition 3. 2, p. 132. For more details see [3]. Section 1.

Proposition 2. Let R be an order in a simple Artinian ring Q and let I be a fractional right R-ideal. Then I is projective iff $II^* = O_l(I)$, and in this case I and I^* are finitely generated R-modules.

We now shift our attention to hereditary Noetherian prime rings.

Theorem 3. Let R be an hereditary Noetherian prime ring which is not simple Artinian, and let $J \subseteq I$ be right ideals of R. Then I/J is an Artinian R-module iff J is an essential submodule of I.

Proof. If J is not essential in I. there is a right ideal $K \subseteq I$ such that $J \oplus K \subseteq I$. If I/J were Artinian, K would contain a minimal right ideal, a contradiction, by proposition 1, of the assumption that R is not simple Artinian.

Suppose J is essential in I. There is a right ideal H such that $I \oplus H$ is essential in R; and then $J \oplus H$ is essential in R too and $I \oplus H / J \oplus H \cong I/J$. Hence we may assume that J and I are essential in R to begin with. Suppose

$$I \supseteq I_1 \supseteq I_2 \supseteq \cdots \supseteq J$$

is a decending chain of essential right ideals. Evidently

$$I^* \subseteq I_1^* \subseteq I_2^* \subseteq \cdots \subseteq J^*$$

is an ascending chain of submodules of the finitely generated left module J^* (Proposition 2). Hence the chain terminates.

Set

$$I_i^{**} = \{ q \in Q \mid I_i^* q \subset R \}.$$

Obviously $I_i \subset I_i^{**}$. On the other hand.

$$I_i = I_i R \supseteq I_i I_i^* I_i^{**} \supseteq I_i^{**};$$

since $I_i I_i^* = O_\ell(I) \ni 1$. Thus the original chain terminates. QED.

Lemma 4. Let R be an hereditary Noetherian prime ring. Then any projective module is a direct sum of uniform right ideals.

Proof. In [1] Theorem 5. 3. p. 13 it is shown that if $R = \oplus_{i=1}^{n} I_i$ is a direct sum of right ideals, then any projective module has the form $\bigoplus_{\ell \in k} J_k$, where each J_k is contained in some I_i. Hence it sffices to show that R is a finite direct sum of uniform right ideals. But we have already remarked that each maximal uniform right ideal U is the right annhilator of some element $r \in R$. Since rR is projective, the exact sequence

$$O \to U \to R \to rR \to O$$

splits. Iterating the argument, we see that R is a direct sum of uniform right ideals. QED.

Corollary 5. Let R be an hereditary Noetherian prime ring, I and I' two right ideals of the same uniform dimension. Then I is isomorphic to an essential submodule of I.′

Proof. It suffices to note that I and I′ are direct sum of uniform right ideals of the same length, and that each uniform right ideal is isomorphic to an essential submodule of every other uniform right ideal. QED.

References

1. H. Cartan and S. Eileberg, "*Homological Algebra*," princeton Univ. press, princeton, N. J., 1956.
2. A. W. Goldie, "*Rings with Maximum condition*," mimeographed notes, Yale Univ., 1961.
3. J. C. Robson, *Non-commutative Dedekind rings*, J. Algebra 9 (1968), 249–265.

Seoung Jun University