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SOME CHARACTERIZATIONS ON REAL HYPERSURFACES OF A COMPLEX PROJECTIVE SPACE

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§ 0. Introduction

Recently many authors studied sufficient conditions and necessary and sufficient condition for a real hypersurface of a complex m-dimensional complex projective space CP^m to be one of model spaces $M^{C}_{p,q}(a,b)$ by using the theory of Riemannian submersion (*cf*. Ki [2], Kim [2], Kon [8], Lawson [3], Maeda [4], Okumura [5], Pak [2, 6], Takagi [7], Yano [8] etc.).

The model space $M^{C}_{p,q}(a,b)$ above mentioned is discribed in the following.

Let S^{2m+1} be an odd-dimensional unit sphere of curvature 1. We consider the natural projection $\tilde{\pi}$ of S^{2m+1} onto CP^m that is defined by the Hopf-fibration

$$S^1 \to S^{2m+1} \to CP^m,$$

which is the Riemannian submersion with totally geodesic fibres.

Let $\bar{M}$ and M be Riemannian manifolds of dimension $2m$, $2m-1$ respectively and $\pi: \bar{M} \to M$ be a differentiable map. $(\bar{M}, M, \pi)$ is called a Riemannian submersion compatible with the Hopf-fibration if the following conditions are satisfied.

(1) $\bar{M}$ and M are (real) hypersurfaces of S^{2m+1} and CP^m respectively.

(2) $\pi: \bar{M} \to M$ is a Riemannian submersion with totally geodesic fibres such that the following diagram commutes:

$$\begin{array}{ccc} \bar{M} & \xrightarrow{\quad \tilde{i} \quad} & S^{2m+1} \\ \pi \downarrow & i & \downarrow \tilde{\pi} \\ M & \longrightarrow & CP^m \end{array}$$

where $\tilde{i}$ and i denote the immersions in (1).

We denote by $S^{2p+1}(a)$ the hypersphere of radius a centered at the origin in C^{p+1}. If we identify C^{p+q+2} with the product space $C^{p+1} \times C^{q+1}$, then, tak-

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ing spheres $S^{2p+1}(a)$ in C^{p+1} and $S^{2q+1}(b)$ in C^{q+1}, we can consider the product space $\bar{M}^C_{p,q}(a, b)=S^{2p+1}(a) \times S^{2q+1}(b)$ as a submanifold in C^{p+q+2}. Thus, if $a^2+b^2=1$, for any portion (p, q) of $m-1$, $p\geq 0$, $q\geq 0$, $\bar{M}^C_{p,q}(a, b)$ may be considered as a real hypersurface of S^{2m+1}. Putting $M^C_{p,q}(a, b)=\tilde{\pi}(M^C_{p,q}(a, b))$. we get a Riemannian submession $(\bar{M}^C_{p,q}(a, b), M^C_{p,q}(a, b), \pi)$ compatible with the Hopf-fibration.

$M^C_{p,q}(a, b)$ thus obtained has several characteristic properties, which can be used to prove M to be congruent to $M^C_{p,q}(a, b)$ for some p, q Maeda [4] and Okumura [5] proved the following theorem which gives several characterizations of $M^C_{p,q}(a, b)$.

Theorem A. *Let M be a real hypersurface of CP^m and $\pi: \bar{M}\to M$ the submersion compatible with the Hopf-fibration $\tilde{\pi}: S^{2m+1}\to CP^m$. Then the following conditions* (1)~(3) *are equivalent to each other* :

(1) *The second fundamental tensor of $\bar{M}$ is parallel.*

(2) *The contact structure of M induced from CP^m commutes with the second fundamental tensor A of M.*

(3) *The derivative of A has the contact length equal to $\|\nabla A\|^2=4(m-1)$. Moreover, if one of the above condition* (2) *and* (3) *is valid, then $M=M^C_{p,q}(a, b)$ for some portion (p, q) of $m-1$ and some a, b such that $a^2+b^2=1$.*

The equivalence of the condition (1), (2) and (3) in Theorem *A* was proved by Pak [6], one of the present authors.

When $\bar{M}$ is I) an Einstein space or II) a locally symmetric space, it is well known that $\bar{M}$ has parallel second fundamental tensor. Projecting the quantities on $\bar{M}$ onto M in CP^m, a further observation on $M^C_{p,q}(a, b)$ can be made. In fact Maeda [4] also studied the hypersurface with conditions corresponding to I) or II) by using Theorem *A*.

The purpose of the present paper is to study necessary and sufficient conditions for M to be one of model spaces $M^C_{p,q}(a, b)$ which are also concerned with the locally symmetry of $M^C_{p,q}(a, b)$.

Manifolds, hypersurfaces, geometric objects and mappings we discuss in this paper will be assumed to be differenciable and of class C^∞. Throughout the paper we use system of indices as follows :

$\kappa, \mu, \nu, \lambda=1, 2, \cdots, 2m+1$; $h, i, j, k, s, t=1, 2, \cdots, 2m$,

$\alpha, \beta, \gamma, \delta=1,2,\cdots,2m;\quad a, b, c, d, e, r=1,2,\cdots 2m-1.$
The summation convention will be used with respect to these system of indices.

§ 1. Real hypersurface of CP^m.

It is well known that the complex projective space CP^m is a Kaehlerian manifold of constant holomorphic sectional curvature 4. Let CP^m be covered by a system of coordinate neighborhoods $\{U; y^h\}$ and F_i^h components of the Kaehlerian structure tensor and g_{ji} those of the Hermitian metric tensor of CP^m. Then we have

$$(1.1)\qquad F_j^i F_i^h=-\delta_j^h,\qquad F_j^s F_i^t g_{ts}=g_{ji}$$

Denoting by D_j the operator of covariant differentiation with respect to g_{ji}, we get

$$(1.2)\qquad D_j F_i^h=0$$

Let's denote by $K_{kji}{}^h$ components of the curvature tensors of CP^m. Then we have

$$(1.3)\qquad K_{kji}{}^h=\delta_k^h g_{ji}-\delta_j^h g_{ki}+F_k^h F_{ji}-F_j^h F_{ki}-2F_{kj}F_i^h$$

because CP^m is of constant holomorphic sectional curvature 4.

We consider a real hypersurface M of CP^m covered by a system of coordinate neighborhoods $\{U; x^a\}$ and immersed isometrically by the immersion $i: M\to CP^m$ represented by $y^j=y^j(x^a)$. In a neighborhood of each point, we denote by B_a^h the vectors $\partial_a y^j$ ($\partial_a=\partial/\partial x^a$) tangent to M and choose a unit normal vector field N^i in CP^m. Then $g_{cb}=g_{ji}B_c^j B_b^i$ are components of the induced metric tensor in M. Denoting by ∇_b the operator of the van der Waerden-Bortolotti covariant differentiation with respect to g_{ba}, equations of Gauss and Weingarrten are given by

$$(1.4)\qquad \nabla_b B_a^h=A_{ba}N^i,\qquad \nabla_b N^i=-A_b^a B_a^i$$

respectively, A_{ba} being the second fundamental tensor with respect to the normal vector N^i, $A_b^a=A_{bc}g^{ca}$, where $(g^{ba})=(g_{ba})^{-1}$.

On the other hand, in each coordinate neighborhood U we can put

$$(1.5)\qquad F_i^h B_a^i=f_a^b B_b^h+u_a N^h,\qquad F_i^h N^i=-u_a B_a^h,$$

f_a^b being a local tensor field of type (1, 1) and u_a a local 1-form defined in U, where $u^a=g^{ab}u_b$.

If we apply F_h^j to (1.5) and use (1.1) and (1.5) itself, we can easily obtain

(1.6) $f_e^b f_a^e = -\delta_a^b + u_a u^b$, $u_e f_a^e = 0$, $f_e^a u^e = 0$, $u_e u^e = 1$,

which means that the aggregate (f_a^b, u_a, u^b) is an almost contact structure.

Applying the operator ∇_c to (1.5) and using (1.2), (1.4) and (1.5), we can also find

(1.7) $\nabla_c f_a^b = A_c^b u_a - A_{ca} u^b$, $\nabla_c u_a = -A_{ce} f_a^e$

Denoting by $K_{dcb}{}^a$ components of the curvature tensor with respect to the induced metric g_{ba} in M, we have from (1.3) that equations of Gauss and Codazzi are given by

(1.8) $K_{dcb}{}^a = \delta_d^a g_{cb} - \delta_c^a g_{db} + f_d^a f_{cb} - f_c^a f_{db} - 2 f_{dc} f_b^a + A_d^a A_{cb} - A_c^a A_{db}$,

(1.9) $\nabla_c A_{ba} - \nabla_b A_{ca} = u_c f_{ba} - u_b f_{ca} - 2 f_{cb} u_a$

From now on we prepare several fundamental lemmas for later use.

Lemma 1. (See also [4]). *Let M be a real hypersurface of CP^m and assume that the induced vector field u^a is a principal curvature vector with principal curvature λ, or simply a P. C. vector with P. C. λ. Then*

(1) *λ is locally constant,*

(2) *if X^a is a P. C. vector with P. C. γ and orthogonal to u^a, $f_e^a X^e$ is a P. C. vector with P. C. $(\lambda\gamma + 2)(2\gamma - \lambda)$.*

Proof) If the induced vector field u^a is a *P. C.* vector, then we can see from (1.6) and (1.7) that the trajectories of u^a are geodesics, *i. e.*, $u^e \nabla_e u = 0$ because u^a is a unit vector. Hence the *P. C.* λ is given by $\lambda = A_{ba} u^b u^a$, *i. e.*,

Applying the operator ∇_c to (1.10) and using (1.7), we have

(1.11) $(\nabla_c A_{ba}) u^a + A_{ba} A_c^e f_e^a = (\nabla_c \lambda) u_b - \lambda A_{ce} f_b^e$,

from which, taking the skew-symmetric part and using (1.6) and the equation (1.9) of Codazzi,

(1.12) $-2 f_{cb} - 2 A_{ca} A_b^e f_e^a = (\nabla_c \lambda) u_b - (\nabla_b \lambda) u_c - \lambda (A_{ce} f_b^e - A_{be} f_c^e)$.

Transvecting (1.12) with u^c and using (1.6) and (1.10), we obtain

(1.13) $\nabla_b \lambda = \beta u_b$, $\beta = u^e \nabla_e \lambda$,

which together with (1.12) gives

(1.14) $2 f_{cb} + 2 A_{ca} A_b^e f_e^a = \lambda (A_{ce} f_b^e - A_{be} f_c^e)$.

Hence the assertion (2) immediately follows.

Next applying the operator ∇_c to (1.13) and using (1.7), we have

$$\nabla_c \nabla_b \lambda = (\nabla_c \beta) u_b - \beta A_{ce} f_b^e,$$

from which, taking the skew-symmetric part,

$$(\nabla_c \beta) u_b - (\nabla_b \beta) u_c - \beta (A_{ce} f^e_b - A_{be} f^e_c) = 0$$

and consequently

$$\beta (A_{ce} f^e_b - A_{be} f^e_c) = 0$$

because of $\nabla_b \beta = (u^e \nabla_e \beta) u_b$. *If* $\beta \neq 0$ *at a point, then* $A_{ce} f^e_b = A_{be} f^e_c$ *at that point. Thus, if* X^a *is a P. C. vector with P. C.* γ, $f^a_e x^e$ *is a P. C. vector with P. C.* $-\gamma$, *which contradicts the assertion* (2). *Henceg* $\beta = 0$, *which together with* (1.13) *implies the assertion* (1).

At each point of M, we can take orthonormal vectors u^a, X^a_x, $f^a_e X^e_x$ ($x = 1, 2, \cdots, m-1$) which are *P. C.* vectors. Then any tangent vector can be expressed in the form:

$$Y^a = \alpha u^a + \sum_{x=1}^{m-1} \beta^x X^a_x + \sum_{x=1}^{m-1} \gamma^x f^a_e X^e_x .$$

Using the above expression of Y^a, we obtain

Lemma 2. (See also [4]). *Let M be a real hypersurface of* CP^m *and assume that the induced vector field* u^a *is a P. C. vector. For any P. C. vector* Y^a *with P. C.* γ, *if* $f^a_e Y^e$ *is also a P. C. vector with P. C.* γ, *then the contact structure of M induced from* CP^m *commutes with the second fundamental tensor of M. i. e.,*

$$A_{ce} f^e_b + A_{be} f^e_c = 0$$

at every point of M.

§ 2. Some properties concerning with locally symmetry.

Let $(\tilde{M}, M, \pi)$ be a Riemannian submersion compatible with the Hopf-fibration $\tilde{\pi} : S^{2m+1} \to CP^m$.

Covering S^{2m+1} by a system of coordinate neighborhoods $\{ \hat{U}; y^\kappa \}$ such that $\tilde{\pi}(\hat{U}) = \tilde{U}$ are coordinate neighborhoods of CP^m with local coordinate (y^j), we can represent the projection $\tilde{\pi}$ by $y^j = y^j(y^\kappa)$ and put $E^j_\kappa = \partial_\kappa y^j$ ($\partial_\kappa = \partial / \partial y^\kappa$) with the rank of the matrix (E^j_κ) being always $2m$.

Let's denote by $\tilde{\xi}^\kappa$ components of $\tilde{\xi}$ the unit Sasakian structure of S^{2m+1} induced from C^{m+1}. Then $\{ E^j_\kappa, \tilde{\xi}_\kappa \}$ is a local coframe in $\hat{U}$, where $\tilde{\xi}_\kappa = \tilde{\xi}^\mu g_{\mu\kappa}$, $g_{\mu\kappa}$ being components of the metric tensor on S^{2m+1}. If we define E^κ_j by $(E^\kappa_j, \tilde{\xi}^\kappa) = (E^j_\kappa, \tilde{\xi}_\kappa)^{-1}$, then $\{ E^\kappa_j, \tilde{\xi}^\kappa \}$ is a local frame in $\hat{U}$, which is dual to $\{ E^j_\kappa, \tilde{\xi}_\kappa \}$.

We now take coordinate neighborhoods $\{\bar{U}; x^\alpha\}$ of $\bar{M}$ such that $\pi(\bar{U})=U$ are coordinate neighborhoods of M with local coordinates (x^a). Denoting the immersion $\tilde{i}$ and the submersion π by $y^\kappa=y^\kappa(x^\alpha)$ and $x^a=x^a(x^\alpha)$ respectively, the commutativity $\tilde{\pi}\circ\tilde{i}=i\circ\pi$ implies

$$B_a^j E_\alpha^a = E_\kappa^j B_\alpha^\kappa, \quad E_j^\kappa B_b^j = B_\beta^\kappa E_b^\beta,$$

where $E_\alpha^a=\partial_\alpha x^a$ and $B_\alpha^\kappa=\partial_\alpha y^\kappa$. Hence the structure vector $\tilde{\xi}$ is always tangent to $\bar{M}$. If we denote by ξ^α components of $\tilde{\xi}$ with respect to the coordinates (x^α), we can similarly get a local frame $\{E_a^\alpha, \xi^\alpha\}$ and its dual coframe $\{E_\alpha^a, \xi_\alpha\}$ in $\bar{U}$, where ξ_α is the associate vector field of ξ^α with respect to the metric tensor $g_{\beta\alpha}=g_{\mu\kappa}B_\beta^\mu B_\alpha^\kappa$ of M.

Since the metrics $g_{\lambda\mu}$ on S^{2m+1} and $g_{\beta\alpha}$ on $\bar{M}$ are invariant with respect to the submersions $\tilde{\pi}$ and π respectively, the van der Waerden-Bortolotti covariant derivatives of E_λ^i, E_i^λ and E_α^a, E_a^α are respectively given by (*cf.* [1])

(2.1) $\tilde{D}_\mu E_\lambda^i = -F_j^i(E_\mu^j\tilde{\xi}_\lambda+\tilde{\xi}_\mu E_\lambda^j)$, $D_\mu E_i^\lambda = -F_{ji}E_\mu^j\tilde{\xi}^\lambda+F_i^j\tilde{\xi}_\mu E_j^\lambda$,

(2.2) $\nabla_\beta E_\alpha^a = -f_b^a(E_\beta^b\xi_\alpha+\xi_\beta E_\alpha^b)$, $\nabla_\beta E_a^\alpha = -f_{ba}E_\beta^b\xi^\alpha+f_a^b\xi_\beta E_b^\alpha$,

where $\tilde{D}_\mu$ and ∇_β denote the operators of covariant differentiations with respect to $g_{\lambda\mu}$ and $g_{\beta\alpha}$ respectively.

Moreover, equations of co-Gauss, of co-Codazzi and of co-Ricci for the submersion π are respectively given by (*cf.* [1])

(2.3) $$K_{dcb}{}^a = K_{dcb}{}^a - f_d^a f_{cb} + f_c^a f_{db} + 2f_{dc}f_b^a,$$

(2.4) $$K_{dcb} = -\nabla_d f_{cb} + \nabla_c f_{db},$$

(2.5) $$K_{ocb}{}^a = f_{ce}f_b^e,$$

where $K_{dcb}{}^a = K_{\delta\gamma\beta}{}^\alpha E_d^\delta E_c^\gamma E_b^\beta E_\alpha^a$, $K_{dcb} = K_{\delta\gamma\beta}{}^\sigma E_d^\delta E_c^\gamma E_b^\beta \xi_\sigma$ and $K_{ocb}{}^a = K_{\delta\gamma\beta}{}^\sigma \xi^\delta E_c^\gamma E_b^\beta \xi_\sigma$, $K_{\delta\gamma\beta}{}^\alpha$ being components of the curvature tensor of $\bar{M}$.

Hence, if $\bar{M}$ is a locally symmetric space, applying the operator $\nabla_c=E_c^\alpha\nabla_\alpha$ to (2.3)~(2.5) and using (1.8) and (2.2)~(2.5), we can easily obtain

(2.6) $$f_d^e K_{ecb}{}^a + f_c^e K_{deb}{}^a + f_b^e K_{dce}{}^a + f_e^a K_{dcb}{}^e = 0,$$

(2.7) $$(f_{ed}u_b+f_{eb}u_d+\nabla_e A_{db})A_{ca}+(f_{ec}u_a+f_{ea}u_c+\nabla_e A_{ca})A_{db}$$
$$-(f_{ed}u_a+f_{ea}u_d+\nabla_e A_{da})A_{bc}-(f_{ec}u_b+f_{eb}u_c+\nabla_e A_{cb})A_{da}=0,$$

(2.8) $$f_{de}u_c u_b - f_{ce}u_d u_b + f_{ae}(A_d^a A_{cb}-A_c^a A_{db}) = \nabla_e\nabla_d f_{cb} - \nabla_e\nabla_c f_{db},$$

(2.9) $$f_d^c A_a^d u^a = 0$$

As already mentioned in § 0, Maeda [4] proved that a complete real hypersurface M satisfying the conditions (2.6) and (2.9) is congruent to $M_{p,q}^{C}(a, b)$.

On the other hand, if M has parallel second fundamental tensor, M satisfies (2.7) and (2.9) because of Theorem A. From this point of view, the necessity of a further observation on $M^{S}_{p,q}(a, b)$ will be naturally occured.

§3. Real hypersurfaces satisfying (2.7) and (2.9).

Let M be a real hypersurface of CP^m which satisfies the conditions (2.7) and (2.9).

We first notice that (2.9) implies

$$(3.1) \qquad A_{ba}u^a = \lambda u_b, \qquad \lambda = A_{ba}u^b u^a,$$

i. e., that the induced vector field u^a is a *P. C.* vector with *P. C.* λ and consequently λ is locally constant by means of Lemma 1. Thus (1.11) with $\lambda =$ const. gives

$$(3.2) \qquad (\nabla_c A_{ba})u^a = -\lambda A_{ce} f^e_b - A_{ba} A^e_c f^a_e.$$

Next, transvecting (2.7) with $u^c u^a$ and substituting (3.2), we can find

$$\lambda \{ \nabla_c A_{ba} + \lambda A_{ce} f^e_a u_b + A^e_c f^d_e A_{da} u_b + \lambda A_{ce} f^e_b u_a + A^e_c f^d_e A_{db} u_a \} = 0,$$

from which, if $\lambda \neq 0$, we have

$$\nabla_c A_{ba} = -\lambda A_{ce} (f^e_b u_a + f^e_a u_b) - A^e_c (f^d_e A_{bd} u_a + f^d_e A_{ad} u_b),$$

from which, taking skew-symmetric part with respect to c and b and using the equation (1.9) of Codazzi, we find

$$u_c f_{ba} - u_b f_{ca} - 2 f_{cb} u_a = -\lambda A_{ce} (f^e_b u_a + f^e_a u_b) + \lambda A_{be} (f^e_c u_a + f^e_a u_c) - A^e_c f^d_e (A_{bd} u_a + A_{ad} u_b) + A^e_b f^d_e (A_{ca} u_a + A_{ad} u_c),$$

from which, together with the skew-symmetry of f_{ca}, gives

$$(3.3) \qquad A^e_c f^a_e = f^e_c A^a_e$$

because we assumed that $\lambda \neq 0$.

Now we suppose that $\lambda = 0$. Differentiating (2.7) covariantly and transvecting thus obtained equation with u^c, we can easily verify that

$$(3.4) \quad (\overset{*}{\nabla}_e A_{db}) f_{ca} - (\overset{*}{\nabla}_e A_{da}) f_{cb} + \{ -2 A_{ce} u_a + A_{ca} u_e + (\nabla_c \nabla_e A_{ra}) u^r \} A_{db} - \{ -2 A_{ce} u_b + A_{cb} u_e + (\nabla_c \nabla_e A_{rb}) u^r \} A_{da} = 0$$

with the help of (1.6) (1.7) and (1.14) with $\lambda = 0$, where we have put $\overset{*}{\nabla}_c A_{ba} = \nabla_c A_{ba} + f_{cb} u_a + f_{ca} u_b$. Taking the skew-symmetric part of (3.4) with respect to e and c, we obtain

$$(3.5) \quad (\overset{*}{\nabla}_e A_{db}) f_{ca} - (\overset{*}{\nabla}_c A_{db}) f_{ea} - (\overset{*}{\nabla}_e A_{da}) f_{cb} + (\overset{*}{\nabla}_c A_{da}) f_{eb} = 0$$

bacause $(\nabla_d \nabla_c A_{ba} - \nabla_c \nabla_d A_{ba}) u^b = -A_{da} u_c + A_{ca} u_d$ which is a direct consequence of the equation (1.8) of Gauss and (3.1) with $\lambda=0$.

Transvecting (3.5) with f^{ca} and using (1.6), we get

(3.6) $$(2m-4)\overset{*}{\nabla}_e A_{db}+f_{ed}u_b+(\nabla_e A_{da})u^a u_b+\{(2m-2)u_d+(\nabla_c A_{da})f^{ca}\}f_{eb}$$
$$+u_e u^c\nabla_c A_{db}=0,$$

from which, transvecting with u^e together with $u^e(\nabla_e A_{da})u^a=0$,

$$(2m-3)u^c\nabla_c A_{db}=0.$$

On the other hand, as a direct consequence of the equation (1.9) of Codazzi, we easily see that

$$(\nabla_c A_{da})f^{ca}=\frac{1}{2}(\nabla_c A_{da}-\nabla_a A_{dc})f^{ca}=-(2m-2)u_d.$$

Therefore, (3.6) reduces to $\overset{*}{\nabla}_e A_{db}=0$, provided $m\geq 3$, which gives $\|\nabla_c A_{ba}\|^2=4(m-1)$ and consequently (3.3) follows by means of Theorem *A*. Hence, in any case either $\lambda\neq 0$ or $\lambda=0$, (3.3) is valid at any point of *M*, which and Theorem *A* give

Theorem 1. *$M^c_{p,q}$ (a, b) are only complete real hypersurfaces of $CP^m(m\geq 3)$ satisfying* (2.7) and (2.9).

§4. Real hypersurfaces satisfying (2.8).

It is well known that, if $\bar{M}$ is a locally symmetric or Einstein space, $\bar{M}$ has parallel second fundamental tensor and consequently the mean curvature of *M* is constant. From this point of view, in this section we shall determine real hypersurfaces of CP^m which satisfying (2.8) and have constant mean curvature.

We first prove

Lemma 3. *Let M be a real hypersurface of CP^m which satisfies* (2.8) *and has constant mean curvature. Then the induced vector field u^a is a P. C. vector with P. C. $\lambda=A_{ba}u^b u^a$.*

Proof) Using (1.7), the condition (2.8) can be written by

(4.1) $$(A^e_c f_{ae}+A^e_a f_{ce})A_{db}-(A^e_d f_{ae}+A^e_a f_{de})A_{cb}$$
$$=(\nabla_a A_{db})u_c-(\nabla_a A_{cb})u_d+(f_{ad}u_c u_b+f_{ca}u_d u_b),$$

from which, transvecting with u^a and using (1.6), we have

$$(f^a_c A_{ae}u^e)A_{db}-(f^a_d A_{ae}u^e)A_{cb}=(u^e\nabla_e A_{db})u_c-(u^e\nabla_e A_{cb})u_d.$$

Transvecting the above equation with f^{db}, we find

(4.2) $$A_{ca}(A_b^a u^e)=\lambda A_{cb}u^b$$

Transvecting also (4.1) with f^{cb}, we can get

$$A_c^e A_{eb}=(A_{ce}u^e)(A_{bd}u^d)-f^{ed}A_{db}f_{ca}A_e^a,$$

from which, taking skew-symmetric part,

$$f^{ed}A_{ea}(A_{db}f_c^a-A_{dc}f_b^a)=0$$

and consequently $f_b^a A_a^e f_e^b A_{dc}u^c=0$, Thus, transvecting this equation with f_r^b implies

(4.3) $$A_c^e f_e^d A_{db}u^b=0$$

because of $f_{cb}(A_e^c u^e)(A_d^b u^d)=0$.

Next we contract (4.1) with respect to c and b. Then we have by using the assumption $A_e^e=A_{cb}g^{cb}=$ const. and (1.6)

$$(f_a^e A_e^c-A_a^e f_e^c)A_{dc}-A_e^e(f_a^e A_{ed}-A_a^e f_{ed})=(\nabla_a A_{dc})u^c+f_{ad},$$

from which, taking skew-symmetric part and using the equation (1.9) of Codazzi,

(4.4) $$(A_{ec}f_a^e+A_{ea}f_c^e)A_d^c-(A_{ec}f_d^e+A_{ed}f_c^e)A_a^c=0.$$

Transvecting (4.4) with u^d and using (4.2) and (4.3), we find $\lambda f_a^e A_{ec}u^c=0$, from which, applying f_b^a gives $\lambda\{A_{be}u^e-\lambda u_b\}=0$. If we put the set $S=\{p\in M \mid (A_{be}u^e-\lambda u_b)_p\neq 0\}$, then $A_{be}u^e\neq 0$ on S because $\lambda=0$ on S.

On the other hand, it follows from (4.2) that $A_{be}u^e=0$ on S. It contradicts and hence S must be void. Thus $A_{be}u^e=\lambda u_b$ at any point of M.

Lemma 4. *Under the same assumptions as stated in Lemma 3, the contact structure of M induced from CP^m commutes with the second fundamental tensor.*

Proof) Since u^a is a $P.C.$ vector by means of Lemma 3, if we transvect (4.1) with $u^c u^b$, we find

$$-\lambda(A_{ae}f_d^e+A_{de}f_a^e)=f_{ad}+(\nabla_a A_{db})u^b,$$

from which, taking symmetric part,

(4.5) $$-2\lambda(A_{ae}f_d^e+A_{de}f_a^e)=(\nabla_a A_{db})u^b+(\nabla_d A_{ab})u^b.$$

On the other hand, (1.11) with $\lambda=$const. gives

$$(\nabla_a A_{db})u^b+(\nabla_d A_{ab})u^b=-\lambda(A_{ae}f_d^e+A_{de}f_a^e),$$

which together with (4.5) implies

$$\lambda(A_{ae}f_d^e+A_{de}f_a^e)=0.$$

Case I). If $\lambda \neq 0$, the lemma is trivial.

Case II). When $\lambda = 0$, by means of Lemma 1, we can take orthonormal vectors u^a, X_x^a, $f_e^a X_x^e$ $(x = 1, 2, \cdots, m-1)$ which are *P. C.* vectors with *P. C.* λ, γ_x, $1/\gamma_x$ respectively. Applying X_y^a to (4.4), we have

$$(\gamma_x^2 - 1)^2 f_a^b X_y^a = 0,$$

i. e., $\gamma_x = 1/\gamma_x$ $(x = 1, 2, \cdots, m-1)$, which and Lemma 2 imply our lemma.

Thus, combinning Theorem A and Lemma 4, we have

Theorem 2. *Let M be a complete real hypersurface of* CP^m *which satisfies* (2.8) *and has constant mean curvature. Then M is congruent to* $M_{p,q}^C(a, b)$ *for some portion* (p, q) *of* $m-1$ *and some a, b such that* $a^2 + b^2 = 1$.

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