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Some Properties on Noetherian Rings

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§ 1. Introduction

Recently, the theory of Commutative rings is blindingly developed together with algebraic geometry ([1], [8], [10], [12], [14]).

In particular, the concepts of height, dimension and depth have been studied deeply ([2], [3], [4], [9], [13], [15]). The purpose of this paper is to prove some properties on noetherian rings. (Proposition 2. 9, Proposition 3. 1, and Proposition 3. 4) and to prove our main theorems (Theorem 5. 1 and Theorem 5. 2) which say that for a noetherian local ring$(A, \mathfrak{m})$, a finite A-module M we have

$$Proj.\, dim(M) + depth(M) = depth(A)$$

and for a Cohen-Macaulay A-module M and a element $p \in Ass(M)$ we have

$$Proj.\, dim(M) = ht(p).$$

In details, we shall describe definitions of concepts which are used throughout this paper in section § 2, and also prove Proposition 2. 9 with respect to minimal homomorphisms. In section § 3, we shall prove two propositions with respect to depths and dimensions (Proposition 3. 1 and Proposition 3. 4). In section § 4, we shall define Cohen-Macaulay module and show that some properties on the dimension and depths.

Finally, in section § 5, main theorems of this paper (Theorem 5. 1, Theorem 5. 2) will be proved.

§ 2. Preliminaries

Throughout this paper, all rings are commutative ring with identity. Let A be a ring. By a finite A-module B we mean that B is a finitely generated A-module.

Definition 2. 1 Let φ: $A \to B$ be a ring homomorphism. If B is flat as an A-module then φ is called a flat homomorphism.

For a ring homomorphism $\varphi: A \to B$, there is the natural map ${}^a\varphi$: $Spec(B) \to Spec(A)$, where for each element $p \in Spec(B)$,

$${}^a\varphi(p) = \varphi^{-1}(p) = p \cap A.$$

Lemma 2.2 Let $\varphi: A \to B$ be a ring homomorphism. Then the following are equivalent.

(i) φ is a flat homomorphism.

(ii) For each $p \in Spec(B)$, B_P is flat over A_p, where $p = p \cap A$

Proof. Let us note the following facts.

(a) For a ring homomorphism $\varphi: A \to B$ and a flat A-module M, $M_{(B)} = M \otimes_A B$ is flat over B.

(b) If S is a multiplicative subset of A, then $S^{-1}A$ is a flat A-module.

(c) For a flat ring homomorphism φ: $A \to B$, a flat B-module is flat o· A.

(i) ⇒ (ii)

Consider the canonical ring homomorphism $A \to A_p$. Since A is a flat A-module, by (a) $B \otimes_A A_p = B_p$ is flat over A_p. Therefore, the canonical homomorphism $A_p \to B_p$ is a flat homomorphism. Since p is a prime ideal of B, it is also a prime ideal of B_p, by (b) $(B_p)_P = B_P$ is a flat B_p-module. Since $A_p \to B_p$ is flat, it follows from (c) that B_P is a flat A_p-module.

(ii) ⇒ (i)

In general, for a ring homomorphism $A \to B$, which p is an ideal of B, $p = p \cap A$ and an A-module N we have

$$(Tor_i^A(B, N))_p = Tor_i^{A_p}(B_P, N_p) \quad (i = 1, 2, \cdots) \ ([10]).$$

By our assumption,

$$Tor_1^{A_p}(B_p, N_p) = 0$$

So that B_P is a flat A_p-module. Therefore, for all maximal ideals p of B and for all A-module N,

$$Tor_1^{A_p}(B_P, N_p) = (Tor_1^A(B, N))_P = 0$$

This implies that

$$Tor_1^A = (B, N) = 0$$

for all A-module N([1]). And thus B is flat over A.

Lemma 2.3 If $\varphi : A \to B$ is a flat homomorphism, then the going-down theorem holds for φ.

Proof. Suppose p' and p are prime ideals of A with $p' \subset p$.

Assume that P is a prime ideal of B lying over p. By Lemma 2.2, B_P is flat over A_p. Moreover pB_P contains the image of pA_p under the canonical ring homomorphism $A_p \to A_P$, where A_p and B_P are local rings. B_P is a faithfully flat A_p-module.
Hence $Spec(B_P) \to Spec(A_p)$ is surjective. Since $p'A_p$ is a prime ideal of A_p, there is a prime ideal P'^* of B_P which is lying over $p'A_p$. If we put $P' = P'^* \cap B$, then P' is a prime ideal of B lying over p' which is contained in P.

Definition 2.4 Let A be a noetherian ring and M an A-module.

A prime ideal p of A is called by an *associated prime of* M if there is a submodule of M which is isomorphic to A/p as A-modules.

The set of all associated primes of M is denoted by $Ass_A(M)$ (or by $Ass(M)$).

Definition 2.5 For a ring $A(\neq 0)$, a finite sequence of $n+1$ prime ideals $p_0 \supset p_1 \supset \cdots\cdots \supset p_n$ is said to be a *prime chain of length* n.

For each prime ideal p, the height $ht(p)$ of p is defined by

$$ht(p) = \max_n \{ n \mid p = p_0 \supset p_1 \supset \cdots \supset p_n \text{ is a prime chain} \}$$

Therefore, $ht(p) = 0$ iff p is a minimal prime ideal of A.

For each proper ideal I of A, we define $ht(I)$

$$ht(I) = \inf_{I \subset p \in Spec(A)} \{ ht(p) \}$$

The dimension of A (or *Krull dimension of* A) is defined by

$$dim(A) = \sup_{p \in Spec(A)} \{ ht(p) \}$$

Therefore, the dimension of every principal ideal domain always zero or one.

Let M be an A-module. The dimension of M is defined by

$$dim(M) = dim(A/Ann(M)),$$

where $Ann(M) = \{ a \in A \mid aM = 0 \}$.

Lemma 2.6 Let $A(\neq 0)$ be a ring. Then the following hold.

(i) For each $p \in Spec(A)$, $ht(p) = dim(A_p)$.

(ii) For an ideal I of A

$$dim(A/I) + ht(I) \leqq dim(A).$$

Definition 2.7 Let A be a ring and let M be an A-module. An element $a \in A$ is said to be *M-regular* if

$$a: M \longrightarrow M (m \rightsquigarrow am)$$

is injective.

A sequence $\{a_1, a_2, ,,, a_r\}$ of elements of A is called an ***M-regular sequence*** if for each $i(1 \leqq i \leqq r)$, a_i is $M/(a_1M + ,,, + a_{i-1}M)$-regular. Let $\{a_1, a_2, ,,, a_r\}$ be an M-regular sequence and let I be an ideal of A. If each $a_i(1 \leqq i \leqq r)$ is in I, then $\{a_1, a_2, ,,, a_r\}$ is called an *M-regular sequence in I*. Furthermore, if there is no element $b \in I$ such that $\{a_1, a_2, ,,, a_r, b\}$ is an M-regular sequence in I, then $\{a_1, a_2, ,,,, a_r\}$ is called a ***maximal M-regular sequence in I***.

When A is noetherian, for an ideal I of A and a finite A-module M, $depth_I(M)$ is defined by *the length of a maximal M-regular sequence in I*.

Let $(A, \mathfrak{m})$ be a noetherian local ring with its maximal ideal $\mathfrak{m}$.
Then we write $depth(M)$ or $depth_A(M)$ for $depth_{\mathfrak{m}}(M)$.

The following are clear;

If A is noetherian and M is a finite A-module, then

(i) If $a \in I$ is an M-regular element, then

$$depth_I(M/aM) = depth_I(M) - 1$$

(ii) If $(A, \mathfrak{m})$ is a local ring, $depth_{\mathfrak{m}}(M) = 0$ *iff* $\mathfrak{m} \in Ass(M)$.

(iii) For each $p \in Spec(A)$,

$$depth(M_p) = 0 \text{ as } A_p\text{-module iff } pA_p \in Ass_{A_p}(M_p) \text{ iff } p \in Ass(M).$$

(iv) For each $p \in Spec(A)$,

$$depth(M_p) \text{ as } A_p\text{-module} \geq depth_p(M).$$

Definition 2.8 Let $(A, \mathfrak{m}, k)$ be a local ring, where $k = A/\mathfrak{m}$. Let M and N be finite A-modules.

A homomorphism u: $M \to N$ is said to be *minimal* if $u \otimes I_k$; $M \otimes_A k \to N \otimes_A k$ is an isomorphism.

Proposition 2.9 Let $(A, \mathfrak{m}, k)$ be a noetherian local ring. Then the following hold.

(i) The following statements (a) and (b) are equivalent:

(a) M is free.

(b) $Tor_1^A(k, M)=0$.

(ii) Let M and N be finite A-modules. Then a homomorphism u: $M \to N$ is minimal iff u is surjective and $ker(u) \subset \mathfrak{m}M$.

(iii) For each finite A-module M, there exists a minimal homomorphism u: $M \to F$, where F is Free.

(iv) Let $0 \to K \xrightarrow{v} F \xrightarrow{u} M \to 0$ be an exact sequence of A modules. If M is a finite A-module, u is a minimal homomorphism and F and K are free A-modules, then the homomorphism

$$v^*: Ext_A^i(k, M) \to Ext_A^i(k, F) \quad (i=0, 1, , , ,)$$

induced by v is zero.

Proof. (i) (a)$\Rightarrow$(b): Since any free A-module is flat over A, we have $Tor_1^A(k, M)=0$

(b)$\Rightarrow$(a): Since $M/\mathfrak{m}M = M \otimes_A k$ is a vector space over k, there is a base $\{\bar{x}_1, , , , \bar{x}_n\}$ of $M/\mathfrak{m}M$. Let η: $M \to M/\mathfrak{m}M$ be the canonical projection, and let x_i' be an element of M such that

$$\eta(x_i')=\bar{x}_i \text{ for } i=1, 2, , , , n.$$

Then $x_i'=x_i+\alpha_i m_i$ $(i=1, 2, , , , n)$

where $\alpha_i \in \mathfrak{m}$ and $m_i \in M$. And $\{x_1', , , , x_n'\}$ generates M. We define a free A-module F which is generated by $\{e_1, , , e_n\}$, and an A-homomorphism $f: F \to M$ by $f(e_i)=x_i'$. Put

$$K=ker\ f$$

Then we have an exact sequence

$$0 \to K \to F \to M \to 0$$

of A-modules. Therefore, we have the following long exact sequence

$$\cdots \to Tor_1^A(k, M) \to K \otimes_A k \to F \otimes_A k \to M \otimes_A k \to 0$$

By our assumption $Tor_1^A(k, M)=0$, and we have the following exact sequence

$$0 \to K \otimes_A k \to F \otimes_A k \to M \otimes_A k \to 0$$

By our definition of F and f,

$$f\otimes I_k:\ F\otimes_A k\rightarrow M\otimes_A k$$

is an isomorphism. Since

$$dim_k(F\otimes_A k)=dim_k(M\otimes_A k)=n$$

Thus, $K\otimes_A k=0$, that is, $K\otimes_A k=K/\mathfrak{m}K=0$. Since this implies that $K=\mathfrak{m}K$, by the Nakayama's Lemma we have $K=0$. Therefore $F\cong M$ and thus M is a free A-module.

(ii) Suppose that u is a minimal homomorphism. Then we have $u\otimes I_k: M/\mathfrak{m}M\rightarrow N/\mathfrak{m}N$ is an isomorphism and the diagram

$$\begin{array}{ccc} M & \xrightarrow{\ u\ } & N \\ \downarrow & & \downarrow \\ M/\mathfrak{m}M & \xrightarrow{u\otimes I_k} & N/\mathfrak{m}N \end{array}$$

is commutative.

As in the proof of (i), $u\otimes I_k(x+\mathfrak{m}M)=u(x)+\mathfrak{m}N$ for all $x\in M$, u is clearly surjective. Since $ker(u\otimes I_k)=\{x+\mathfrak{m}M\,|\,u(x)+\mathfrak{m}N=0\}=0$, it is clear that $ker(u)\subset\mathfrak{m}M$.

Conversely, assume that u is surjective and $ker(u)\subset\mathfrak{m}M$. Then $u\otimes I_k$ is surjective. We shall claim that $ker(u\otimes I_k)=0$. Suppose $ker(u\otimes I_k)\neq 0$.

Then $x+\mathfrak{m}M\neq 0$ such that $x+\mathfrak{m}M\in ker(u\otimes I_k)$.

Since $ker(u)\subset\mathfrak{m}M$, $x\notin ker(u)$.

Thus $u(x)\neq 0$. And $u(x)+\mathfrak{m}N=0$, $u(x)\in\mathfrak{m}N$.

This is a contradiction.

(iii) Since M is a finite A-module, there is a minimal set $\{x_1, \cdots x_n\}$ which generates M. Let F be a free A-module with a free basis $\{e_1, \cdots e_n\}$. Then the homomorphism

$$\varphi:\ F\rightarrow M\ (\varphi(e_i)=x_i)$$

induces an isomorphism

$$F/\mathfrak{m}F\cong M/\mathfrak{m}M$$

Since $dim_k(F/\mathfrak{m}F)=dim_k(M/\mathfrak{m}M)=n$.

Therefore φ is a minimal homomorphism.

(iv) Since u is a minimal homomorphism, $ker(u)\subset\mathfrak{m}F$ by (ii).

The exact sequence implies that $K\cong ker(u)$ and so $K\subset\mathfrak{m}F$. For any $f\in Hom_A(k,\mathfrak{m}F)$, and for any $a+\mathfrak{m}\in k$ we have

$$f(\bar{a})=f(a+\mathfrak{m})=f(a(1+\mathfrak{m}))=af(1+\mathfrak{m})\in\mathfrak{m}F.$$

So $a\in\mathfrak{m}$, $f(a+\mathfrak{m})=f(0)=0$.

Therefore $Hom_A(k, K)=0$.

We can take injective resolutions of K, F *and* M respectively as the below diagrams:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \xrightarrow{v} & F & \xrightarrow{u} & M & \longrightarrow & 0 \\
 & & \downarrow d_0 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & J_0 & \xrightarrow{v_0} & J_0' & \xrightarrow{u_0} & J_0'' & \longrightarrow & 0 \\
 & & \downarrow d_1 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & J_1 & \xrightarrow{v_1} & J_1' & \xrightarrow{u_1} & J''_1 & \longrightarrow & 0 \\
 & & \downarrow d_2 & & & & & & \\
 & & J_2 & & & & & &
\end{array}$$

where the vertical sequence are injective resolutions.

Applying the covariant functor $Hom_A(k, -)$ above diagram, we can obtain the induced homomorphism as below:

$$\begin{array}{ccccccc}
0\longrightarrow Hom_A(k, K) & \xrightarrow{d_0^*} & Hom_A(k, J_0) & \xrightarrow{d_1^*} & Hom_A(k, J_1) & \xrightarrow{d_2^*} & Hom_A(k, J_2) \\
\downarrow v^* & & \downarrow v_0^* & & \downarrow v_1^* & & \\
0\longrightarrow Hom_A(k, F) & \xrightarrow{d_0'^*} & Hom_A(k, J_0') & \xrightarrow{d_1'^*} & Hom_A(k, J'_0) & &
\end{array}$$

In case of $n=1$, we have $Ext^1{}_A(k, K)=ker\ d_1^*/im\ d_0^*$.

We claim that $ker\ d_1^* \subset im\ d_0^*$. For any $f\in Hom_A(k, J_0)$ such that $d_1^*(f)=0$, $im(f) \subset ker(d_1)=im\ d_0$.

We can fixed a element $x\in K$ such that $f(1+m)=d_0(x)$.

Now we can define a homomorphism f': $k\to K$ by $f'(1+m)=x$.

Therefore, $f=d_0\circ f'=d_0^*(f')\in im\ d_0^*$. So, $Ert_A{}^1(k, K)=0$.

Similar arguments yields that $Ext_A{}^n(k, K)=0$ *for all* $n\geqq 2$.

§ 3. Some Properties of Depths and Dimensions

Proposition 3.1 Let A and B be noetherian rings. If $\varphi: A\to B$ is a flat homomorphism, then for each ideal I of A and a finite A-module M

$$depth_I(M)=depth_{I_{(B)}}M_{(B)},$$

where $I_{(B)}=I\otimes_A B$ *and* $M_{(B)}=M\otimes_A B$.

In particular, if ${}^a\varphi$: $Spec(B)\to Spec(A)$ is surjective then $ht(I)=ht(IB)$.

Proof. Since B is a flat A-module, we have $I\otimes_A B\cong IB$ which is an ideal of B. Furthermore, we have

$$A_{(B)}/I_{(B)}\cong A/I\otimes_A B\cong B/I_{(B)}\cong B/IB.$$

The first part of our assertion is ascribe to the following two lemmas.

Lemma 3.2 Let φ: $A \to B$ be a flat homomorphism with A noetherian. For each finite A-module M the following holds.

$$Ext_A{}^i(M, N) \otimes_A B = Ext_B{}^i(M_{(B)}, N_{(B)}) \quad (i=0, 1, 2. \cdots)$$

where N is an A-module.

Proof. Since M is an finite A-module, there is a projective resolution

$$X_*: \cdots \to X_n \xrightarrow{d_n} X_{n-1} \to \cdots \to X_1 \xrightarrow{d_1} X_0 \to M \to 0$$

of the A-module M, where each $X_i (i=0, 1, \cdots)$ is finite free A-module.

Let $X_i = A \oplus \cdots \oplus A$ (r-times). Then, the following holds ([7]).

$$Hom_B\ (X_i \otimes_A B,\ N \otimes_A B) \cong Hom_B(B, N \otimes_A B) + \cdots + Hom_B(B, N \otimes_A B)$$
$$\cong N \otimes_A B \oplus \cdots \oplus N \otimes_A B$$

On the other hand,

$$Hon_A(X_i, N) \otimes_A B \cong (Hom_A(A, N) \oplus \cdots \oplus Hom_A(A, N)) \otimes_A B$$
$$\cong (N \oplus \cdots \oplus N) \otimes_A B$$
$$\cong N \otimes_A B \oplus \cdots \oplus N \otimes_A B$$

In consequence,

$$f_i: Hom_B(X_i \otimes_A B,\ N \otimes_A B) \to Hom_A(X_i, N) \otimes_A B$$

is an isomorphism and the following diagram

$$0 \longrightarrow Hom_B(X_0 \otimes_A B,\ N \otimes_A B) \xrightarrow{Hom(d_1 \otimes I_B,\ I_N \otimes I_B)} Hom_B(X_1 \otimes_A B,\ N \otimes_A B) \longrightarrow$$

$$0 \longrightarrow Hom_A(X_0, N) \otimes_A B \xrightarrow{Hom(d_1, I_N) \otimes I_B} Hom_A(X_1, N) \otimes_A B \longrightarrow$$

is commutative.

Since each $X_i \otimes_A B$ is isomorphic to a direct sum of r copies of B, clealy $X_i \otimes_A B$ is a projective B-module.

Furthermore, since B is a flat A-module, the sequence

$$X_* \otimes_A B: \cdots \to X_n \otimes_A B \to \cdots \to X_0 \otimes_A B \to M \otimes_A B \to 0$$

is exact. That is, $X_* \otimes_A B$ is a projective resolution of the B-module $M \otimes_A B$. In consequence, we have

$$Ext_A{}^i(M_{(B)},\ N_{(B)}) = Ext_A{}^i(M, N) \otimes_A B$$

for $i=0, 1, 2, \cdots$

Lemma 3.3 Let A be a noetherian ring. For a finite A-module M, an ideal I of A and an integer $n>0$, the following are equivalent

(i) $Ext_A^i(A/I, M)=0$ *for any* $i<n$.

(ii) There exists an M-regular sequence $\{a_1, a_2, \cdots a_n\}$ of length n in I.

Proof. (i)⇒(ii) Since $Ext_A^0(A/I, M)=0$, we have

$$Hom_A(A/I, M)=0.$$

Assume that every element of I is not an M-regular element.

Then $I\subset \bigcup_{p\in Ass(M)} p$, and thus there exists an element $p\in Ass(M)$ such that $I\subset p$, By Definition 2.4 A/p is isomorphic to a submodule of M, that is, there is a monomorphism $A/p\to M$. Since $I\subset p$, there exists the canonical projection $A/I\to A/p$. Therefore, the composition

$$A/I\longrightarrow A/p\longrightarrow M$$

is not a zero homomorphism. This implies that $Hom_A(A/I, M)\neq 0$ and thus we get a contradiction. Hence, there is a M-regular element a_1 in I.

We put $M_1=M/a_1M$, then

$$0\longrightarrow M\xrightarrow{a_1} M\longrightarrow M_1\longrightarrow 0$$

is a short exact sequence of A-modules.

Since $Ext_A^i(A/I, M)=0$ $(0\leq i<n)$, from the long cohomology exact sequence

$$\cdots\to Ext_A^{i-1}(A/I, M)\to Ext_A^{i-1}(A/I, M)\to Ext_A^{i-1}(A/I, M_1)\to Ext_A^i(A/I, M)\to\cdots$$

we get $Ext_A^i(A/I, M_1)=0$ *for* $i<n-1$. By repeating the above argument we get an M_1-regular element a_2 in I. Therefore, $\{a_1, a_2\}$ is an M-regular sequence, and $Ext_A^i(A/I, M_2)=0$ for $i<n-2$, where

$$M_2=M/a_1M+a_2M.$$

By using this method continuously, we see that there exists an M-regular sequence $\{a_1, a_2, \cdots a_{n-1}\}$, and $Ext_A^0(A/I, M_{n-1})=0$, where

$$M_{n-1}=M/a_1M+\cdots+a_{n-1}M.$$

Therefore, by the same reason as above, there exists an M_{n-1}-regular element a_n in I and thus $\{a_1, a_2, \cdots a_n\}$ is an M-regular sequence in I.

(ii)⇒(i) we shall put

$$M_i=M/a_1M+\cdots+a_{i-1}M$$

Then, from the short exact sequence

$$0\longrightarrow M_{n-1}\xrightarrow{a_n}M_{n-1}\longrightarrow M_n\longrightarrow 0$$

we get the injection

$$0\longrightarrow Hom_A(A/I, M_{n-1})\xrightarrow{a_n^*}Hom_A(A/I, M_{n-1}),$$

where a_n^* is induced by a_n. That is, for each $f\in Hom_A(A/I, M_{n-1})$ and $[a]\in A/I$, $a_n^*(f([a]))=f([a_n a])$. Since $a_n a\in I$, we have $[a_n a]=0$ in A/I and thus a_n^* is a zero homomorphism.

It follows that $Hom_A(A/I, M_{n-1})=Ext_A^0(A/I, M_{n-1})=0$

From the short exact sequence

$$0\longrightarrow M_{n-2}\xrightarrow{a_{n-1}}M_{n-2}\longrightarrow M_{n-1}\longrightarrow 0$$

we get the long exact sequence

$$0\longrightarrow Hom_A(A/I, M_{n-2})\xrightarrow{a_n^*}Hom_A(A/I, M_{n-2})\longrightarrow Hom_A(A/I, M_{n-1})\longrightarrow$$

$$\longrightarrow Ext_A^1(A/IM_{n-2})\xrightarrow{a_{n-1}^*}Ext_A^1(A/I, M_{n-2})$$

By the same reason as above we obtain

$$Hom_A(A/I, M_{n-2})\cong Ext_A^0(A/I, M_{n-2})=0.$$

Since,

$$Hom_A(A/I, M_{n-1})\cong Ext_A^0(A/I, M_{n-1})=0$$

we have the exact sequence

$$0\longrightarrow Ext_A^1(A/I, M_{n-2})\xrightarrow{a_{n-1}^*}Ext_A^1(A/I, M_{n-2})$$

Now we take an injective resolution

$$0\longrightarrow M_{n-2}\longrightarrow I_0\xrightarrow{\delta_0}I_1\xrightarrow{\delta_1}I_2\longrightarrow\cdots$$

of M_{n-2}. Then

$$Ext_A^1(A/I, M_{n-2})=ker(Hom(1_{A/I}, \delta_1))/Im(Hom(I_{A/I}, \delta_0)),$$

where

$$Hom_A(I_{A/I}, \delta_1):\ Hom_A(A/I, I_1)\rightarrow Hom_A(A/I, I_2)$$

and

$$Hom_A(I_{A/I}, \delta_0):\ How_A(A/I, I_0)\rightarrow Hom_A(A/I, I_1).$$

Therefore, for each element $[f]\in Ext_A^1(A/I, M_{n-2})$, we see that $f\in Hom_A(A/I, I_1)$, For each $[a]\in A/I$, Since $a_{n-1}(f(a))=f(a_{n-1}a)=f(0)$, $a_{n-1}a\in I$, and thus $Ext_A^1(A/I, M_{n-2})=0$.

We assume that

$$Ext_A^0(A/I, M_{n-1}) = \cdots = Ext_A^{i-1}(A/I, M_{n-1}) = 0$$

for i $(2<i<n-1)$.

Then the short exact sequence

$$0 \longrightarrow M_{n-i-1} \xrightarrow{a_{n-i}} M_{n-i-1} \longrightarrow M_{n-i} \longrightarrow 0$$

implies that

$$0 \longrightarrow Ext_A^i(A/I, M_{n-i-1}) \xrightarrow{a_{n-i}^*} Ext_A^i(A/I, M_{n-i-1}) \to \cdots$$

is exact. Therefore, we get $Ext_A^i(A/I, M_{n-i-1})=0$ by the above argument. Consequently, we have

$$Ext_A^i(A/I, M)=0$$

for $i=0, 1, \cdots n-1$.

Proof of the Proposition 3.1 (continued):

Since A/I is a finite A-module, by Lemma 3.2 we have

$$Ext_A^i(A/I, M)\otimes_A B \cong Ext_B^i(B/I_{(B)}, M_{(B)}).$$

Thus, if we can prove that

$$(*)\ depth_I(M)=n \Leftrightarrow Ext_A^i(A/I, M)=0 \ for\ i=0, 1, \cdots n-1,$$
$$and\ Ext_A^n(A/I, M)\neq 0,$$

then our first assertion

$$depth_I(M)=depth_{I(B)}(M_{(B)})$$

holds because of that

$Ext_A^i(A/I, M)\otimes_A B=0 \Leftrightarrow Ext_A^i(A/I, M)=0$ by $1\in B$.

In order to prove (*) we shall use mathematical induction on i. If there is no M-reguar element in I, then

$$Hom_A(A/I, M) \cong Ext_A^0(A/I, M)\neq 0$$

as in the proof of (i)⇒(ii) in Lemma 3.3.

Assume that (*) always holds for all $r<n$. *Let* $\{a_1, a_2, \cdots a_n\}$ be a maximal M-regular sequence in I. Then $\{a_2, \cdots a_n\}$ is a maximal M_1-regular sequence in I, where $M_1=M/a_1M$.

By our induction assumption,

$$Ext_A^{n-1}(A/I, M_1)\neq 0.$$

From the following short exact sequence

$$0 \longrightarrow M \xrightarrow{a_1} M \longrightarrow M_1 \longrightarrow 0,$$

we get the long exact sequence for *Ext*

$$0 \to Ext_A^{n-1}(A/I, M_1) \to Ext_A^n(A/I, M) \to Ext_A^n(A/I, M) \to \cdots$$

because of that $Ext_A^{n-1}(A/I, M)=0$, by Lemma 3. 3.

Since $Ext_A^{n-1}(A/I, M_1) \neq 0$, we have

$$Ext_A^n(A/I, M) \neq 0.$$

Hence (*) holds.

We shall prove the second part of our proposition.

By Lemma 2. 3, the going-down theorem holds for the flat homomorphism $\varphi: A \to B$. Therefore, for a $p \in Spec(B)$ *and* $p = p \cap A$ we have the following holds ([6], [10])

$$(**) \quad ht(p) = ht(p) + ht(p/pB).$$

We take a minimal prime over-ideal p of IB such that $ht(p)=ht(IB)$, and put $p=p\cap A$. Then $ht(p/pB)=0$. By (**) we get $ht(p)=ht(p)$. Since $I \subset p$, we have $ht(p) \geqq ht(I)$ and so $ht(IB) \geqq ht(I)$.

Conversely, let p be *a* minimal prime over-ideal of I such that $ht(p)=ht(I)$. We take a prime p of B lying over p (${}^a\varphi$: $Spec(B) \to Spec(A)$ is surjective and the going-down theorem holds for φ).

If necessary, replacing p we may assume that p is a minimal prime over-ideal of pB, that is, $ht(p)=ht(pB)$. Then, again by (**), $ht(p)=ht(p)$, and thus

$$ht(I) = ht(p) = ht(p) \geqq ht(IB)$$

Consequently, $ht(I)=ht(IB)$.

Proposition 3. 4 Let $A=K[X_1, \cdots X_n]$ be a polynomial ring over a field K. Then there exists a subring $B=K[Y_1, \cdots Y_n]$, $Y_i \in K[X_1, \cdots X_n]$, $i=1, 2, \cdots n$ which satisfies the following conitions:

(i) A is integral over B.

(ii) There is no inclusion relation between prime ideals of A lying over a fixed prime ideal of B.

(iii) $dim(A) = dim(B)$.

Proof. We put $Y_1 = f(X) = \sum_{i=1}^{n} a_i M_i(X) \neq 0$ *in* $K[X_1, X_2, \cdots X_n]$, where $0 \neq a_i \in K$ and the $M_i(X)$ are distinct monomials in $X_1, \cdots X_n$ such that $M_i(X) \neq b_i X_1$ for any $b_i \in K$.

For n positive integers $d_1=1$, $d_2, \cdots d_n$ and a monomial $M(X)=\prod_i X_i^{a_i}$ the positive integer $\sum_i a_i d_i$ is called the weight of $M(X)$. By a suitable choice of $d_2, \cdots d_n$, we can make that no two of the monomials $M_1, \cdots M_l$ in $f(X)$ have the same weight ([12]). If we put

$$Y_i = X_i - X_i^{d_i}, \quad (i=2, 3, \cdots n)$$

then $X_i = Y_i + X_i d_i$, and thus

$$Y_1 = f(X) = f(X_1, Y_2 + X_1^{d}, \cdots\cdots Y_n X_1^{d})$$
$$= a_i X_1^{e} + g(X_1, Y_2, \cdots, Y_n),$$

where g is a polynomal over K whose degree in X_1 is less than e and a_i is the coefficient of the term which has the highest weight in $f(X)$. Then we have

$$X_1^{e} + 1/a_i \{g(X_1, Y_2 \cdots, Y_n) - Y_1\} = 0$$

This implies that X_1 is integral over $K[Y_1, \cdots Y_n]$. Since

$$X_i = Y_i + X_i^{d} \ (i=2, 3 \cdots n)$$

we see that $X_2, \cdots X_n$ are integral over $K[Y_1, \cdots Y_n]$. Consequently, $A = K[X_1, \cdots X_n]$ is integral over $K[Y_1, \cdots, Y_n]$. Moreover, since $X_1 \notin K[Y_1, \cdots, Y_n]$ it follows that $K[Y_1, \cdots, Y_n]$ is a proper subring of A. If we put $B = K[Y_1, \cdots Y_n]$ then (i) holds for A and B.

We shall prove that A and B satisfy our condition (ii). In order to do this we need the following ([1], [8], [10] and [12]).

(***) Let B be a subring of a ring A such that

A is integral over B. If B is a local ring and p is the maximal ideal of B, then the prime ideals of A lying over p are precisely the maximal ideals of A.

Since A is integral over B. for each $p \in Spec(B)$ $A_p = A \otimes_B B_p = (B-p)^{-1}A$ is integral over B_p and B_p is contained as a subring. Moreover the prime ideals of A lying over p correspond to the prime ideals of A_p lying over pB_p which are the maximal ideals of A by(***).

Since $B_p \neq 0$, A_p is not zero and it has maximal ideals. For two maximai ideals $\mathfrak{m}_1$ *and* $\mathfrak{m}_2$ of A_p there is no inclusion relation between $\mathfrak{m}_1 \cap A$ and $\mathfrak{m}_2 \cap A$, which are prime ideals of A lying over p Hence condition (ii) is true for A and B. Note that $Spec(A) \to Spec(B)$ is surjective.

Let us prove that $dim(A) = dim(B)$. At first, we shall prove that the going-up theorem holds for $B \subset A$. We take $p \subset p'$ in $Spec(B)$ and p in $Spec(A)$ such that $p \cap B = p$. Then A/p is integral over B/p and it contains B/p as a subring.

Since $Spec(A) \to Spec(B)$ is surjective it also follows that $Spec(A/p) \to Spec(B/p)$ is surjective.

Therefore there exists a prime p'/p lying over p'/p. Then p' is a prime ideal of A lying over p' and thus the going-up theorem holds for $B \subset A$.

Next, we have to note that $B = K[Y_1, Y_2, \cdots Y_n]$ is a noetherian subring of $A = K[X_1, \cdots, X_n]$ which is also a noetherian ring. We shall take $p_1 \subset p_2$ in $Spec(A)$. Then by (ii) there is no inclusion relation between $p_1 \cap B$ and $p \cap B_2$ in $Spec(B)$. This implies that $dim(A) \leq dim(B)$. By the going-up theorem, for $p \subset p'$ in $Spec(B)$ there exist p *and* p' *in* $Spec(A)$ such that $p \subset p'$. This implies that $dim(B) \leq dim(A)$. In consequence, we have $dim(A) = dim(B)$.

§ 4. Some Properties of Cohen-Macaulay modules

Definition 4.1 Let $(A, \mathfrak{m})$ be a noetherian local ring. If $depth(A) = dim(A)$ or $A = 0$, then A is called *a Cohen-Macauly ring*. A finite A-module M is called *a Cohen-Macauly A-module* if $depth(M) = dim(M)$ or $M = 0$. Note that for a non zero finite A-module M $depth(M) \leq dim(M)$ in general.

Lemma 4.2 Let $(A, \mathfrak{m})$ be a noetherian local ring. For a Cohen-Macaulay A-module M the following hold.

(i) For each $p \in Ass(M)$, $dim(A/p) = depth(M)$.

(ii) For each $p \in Spcc(A)$, the A_p-module M_p is a Cohen-Macaulay A_p-module.

Proof. (i) It is well known that

(a) $depth(M) \leq dim(A/p)$ *for all* $p \in Ass(M)$

(b) $Ann(M) = \bigcap_{p \in Ass(M)} p$

Therefore we have

$depth(M) = dim(M) = dim(A/Ann(M)) \geq dim(A/p) \geq depth(M)$. and thus $depth(M) = dim(A/p)$.

(ii) We want to prove that $dim(M_p) = depth(M_p)$. As well-known the following holds:

$$(****) \quad dim\ M_p = ht(p/Ann(M)) \geq depth\ M_p \geq depth_p M.$$

Step 1. We assume that $Ann(M) \subset p$. Then $(A-p) \cap Ann(M) \neq 0$ and thus $M_p = 0$. Therefore, by Definition 4.1, M_p is a Cohen–Macaulay A_p-module.

Step 2. We assume that $Ann(M) \subset p$, we shall prove our assertion by mathematical induction on $depth_p M$.

(a) $depth_p(M)=0$. This case implies that there exists an element $q \in Ass(M)$ such that $p \subset q$, By (i) every element of $Ass(M)$ is a minimal prime over-ideal of $Ann(M)$. (See $dim(A/Ass(M))=dim(A/p)$ as in the proof (i)) It follows that $p=q$, and thus $dim\ M_p=ht(p/Ann(M))=0$. By (****) we have $dim\ M_p=depth(M_p)=0$.

(b) $depth_p(M)>0$ $(M \neq 0)$; we shall at first prove that under our assertion if $a \in A$ is a M-regular element in m then $dim(M/aM)=dim(M)-1$ *and* M/aM is a Cohen-Macaulay A-module. Since a is an M-regular element $Ann(M/aM) \supsetneqq Ann(M)$ and thus $dim(M/aM)<dim(M)$.

On the other hand,

$$Supp(M/aM)=Supp(M) \cap V(a)=V(Ann(M)+aA)$$

where $Supp(M)=\{p \in Spec(A) \mid M_p \neq 0\}$ and

$$V(a)=\{p \in Spec(A) \mid a \in p\}$$

It follows that $dim(M/aM)=dim(Ann(M)+aA) \geqq dim(A/Ann(M))-1=dim(M)-1$, Hence $dim(M/aM)=dim(M)-1$, Let

$$M_1=M/aM.$$

Then $dim(M_1)=dim(M)-1$. On the other hand by Lemma 3.3, We have $depth(M_1)=depth(M)-1$. Since M is a Cohen-Macaulay A-module $depth(M)=dim(M)$ and thus $depth(M_1)=dim(M_1)=dim(M)-1$. That is, M_1 is a Cohen-Macaulay A-module.

In general, for a M-regular sequence $\{a_1 \cdots\cdots a_r\}$ in p, We can prove that $M/a_1M+\cdots+a_rM$ is also a Cohen-Macaulay A-module.

We assume that our assertion holds for $depth_p(M)<n$ and we shall prove our assertion when $n=depth_p(M)$, If $depth_p(M)=n \neq 0$ then there exist an M-regular element a_1 in p. Let us put $M_1=M/a_1M$ For $S=A-p$, Since

$$0 \rightarrow M \xrightarrow{a_1} M(\text{exact}) \Rightarrow 0 \rightarrow S^{-1}M \xrightarrow{a_1} S^{-1}M(\text{exact})$$

the element a_1 is an M_p-regular element in pA_p. Therefore by the proceding statements and Lemma 3.3 we have

$$dim(M_1)_p=dim(M_p/a_1M_p)=dim(M_p)-1,$$
$$depth(M_1)_p=depth(M_p)-1$$

and that M_1 is a Cohen-Macaulay A-module.

Since $depth_p M_1=depth_p M-1<n$ and M_1 is a Cohen-Macaulay A_p-module. by our inductive hypothesis $(M_1)_p$ is a Cohen-Macaulay A_p-module. Therefore we have

i. e $dim(M_p)=depth(M_p)$.

M_p is a Cohen-Macaulay A_p-module.

Lemma 4.3 Let (A, m) be a Cohen-Macaulay ring. Then for every proper ideal I, $ht(I)+dim(A/I)=dim(A)$.

Proof. Since for a minimal prime p over-ideal of I

$$ht(I)=ht(p),\quad dim(A/I)=dim(A/p)$$

If our formula holds for every prime ideal then our Lemma is proved completely. Let p be a prime ideal, and $dim\ (A)=depth\ (A)=n$. By Lemma 4.2, A_p is a Cohen-Macaulay ring, and by (i) of Lemma 2.6, we have

$$depth(A_p)=dim(A_p)=ht(p)$$

Let us put $ht(p)=r$, Hence, we can find an A-regular sequence $\{a_1, \cdots a_r\}$ in p by Lemma 3.3.

As in the proof of Lemma 4.2, $A/(a_1, \cdots, a_r)$ is a Cohen-Macaulay ring with dimension $n-r(dim(A/a_1A)=n-1$, $dim(A_1/a_2A)=dimA_1-1=dimA-2$, where $A_1=A/a_1A$. By repeatting $dim(A/a_1A+\cdots+a_rA)=n-r)$.

Since p is a minimal prime over ideal of $(a_1, \cdots, a_r)$ we have $ht(p)=ht((a_1, \cdots, a_r))$.

Therefore, by (i) of Lemma 4.2, we have

$$dim(A/p)=depth(A/(a_1, \cdots, a_r))=dim(A/(a_1, \cdots, a_r))=n-r$$

$$(note\ that\ p\in Ass(A/(a_1, \cdots, a_r))$$

§ 5. Main Theorems

Let M be a finite A-module. We define that the projective dimension $(Proj.dim(M))$ of M is the length of shortest projective resolution of M.

Theorem 5.1

Let (A, m) be a noetherian local ring and let $M(\neq 0)$ be a finite A-module, then

$$Proj.dim(M)+depth(M)=depth(A).$$

Proof. If A is a noetherian local ring, we know that

M is free $\Leftrightarrow$ M is projective $\Leftrightarrow$ M is flat. ([8], [14])

If M is free then $depth(M)=depth(A)$. and $Proj.dim(M)=0$. Hence our formula is clear.

We will prove our Proposition by mathematical induction on $Proj.dim(M)$.

Case 1. $Proj.dim(M)=1$. In this case there exists a projective resolution;

$$0\to K\xrightarrow{v}F\xrightarrow{u}M\to 0$$

where u is a minimal homomorphism (Definition 2.8), and F, K are free. We assume that $depth(A)=depth(F)=depth(K)=n$. Then, by Lemma 3.3, we have the following:

$$Ext_A{}^i(k, K)=0=Ext_A{}^i(k, F)\ (0\leqq i<n)$$

$$Ext_A{}^n(k, K)\neq 0,\ \ Ext_A{}^n(k, F)\neq 0.$$

where $k=A/\mathfrak{m}$, In the long exast sequence

$$\cdots\to Ext_A{}^i(k, K)\xrightarrow{v^*}Ext_A{}^i(k, F)\longrightarrow Ext_A{}^i(k, M)\longrightarrow Ext_A{}^{i+1}(k, K)\xrightarrow{v^*}\cdots$$

we see $v^*=0$ by (iii) of Proposition 2.9. Therefore

$$Ext_A{}^i(k, M)=0(i<n-1),\ \ Ext_A{}^{n-1}(k, M)\neq 0$$

Hence, by Lemma 3.3, we have depth$(M)=n-1$.

Therefore our formula dolds.

Case 2. $Proj.dim(M)>1$.

Assume that our formula holds for all finite A-module $M(\neq 0)$ with $Proj.dim(M)=r<n$. Let M be a finite A-module with $Proj.dim(M)=r+1$ and consider a projective resolution

$$0\to X_{r+1}\to X_r\to\cdots X_1\xrightarrow{d_1}X_0\to M\to 0$$

of M. We may regard the exact sequence

$$0\to X_{r+1}\to X_r\to\cdots\to X_1\to Imd_1\to 0$$

as a shortest projective resolution of $Im\ d_1$, that is, $Proj.dim(Imd_1)=r$. By our inductive hypothesis $depth(Imd_1)=n-r$,

Therefore, by Lemma 3.3 we have

$$Ext_A{}^i(k, Im\ d_1)=0\ (i\leqq n-r-1),\ \ Ext_A{}^{n-r}(k, Im\ d_1)\neq 0$$

Again, we consider the short exact sequence

$$0\to Im\ d_1\to X_0\to M\to 0$$

We get

$$\cdots\to Ext_A{}^i(k, Im\ d_1)\to Ext_A{}^i(k, X_0)\to Ext_A{}^i(k, M)\to Ext_A{}^{i+1}(k, Im\ d_1)\to\cdots$$

Since $Ext_A{}^i(k, X_0)=0$ *for* $i<n$ we have

$$Ext_A{}^{n-r-1}(k, M)\neq 0,\ \ Ext_A{}^i(k, M)=0\ (i<n-r-1).$$

By Lemma 3.3, we have $depth(M)=n-r-1$.

Hence our formula holds for M.

Now our Proposition holds.

Theorem 5.2.

Let $(A, \mathfrak{m})$ be a Cohen-Macaulay ring and let $M(\neq 0)$ be a Cohen-Macaulay A-module. Then for every $p \in Ass(M)$, we have

$$ht(p) = Proj.dim(M).$$

Proof. By Lemma 4.3 we have

$$ht(p) + dim(A/p) = dim(A)$$

By (i) of Lemma 4.2 we have $dim(A/p) = depth(M)$.

Since

$$Proj.dim(M) + depth(M) = depth(A)$$

by Proposition 4.4, and $dim(A) = depth(A)$ by our hypothesis, we have the following

$$ht(p) + depth(M) = Proj.dim(M) + depth(M).$$

Therefore, we have

$$ht(p) = Proj.dim(M).$$

REFERENCES

1. M. F. Atiyah and I. G. Macdonald, *Introduction to Commutative Algebra,* Addison Wesley, (1969).
2. M. Auslander and D. Buchbaum, *Codimension and Multiplcity,* Ann. of Math. 68, (1958), 625~657.
3. J. E. Carrig, *The homological dimensions of symmetric algebras,* Trans. of A. M. S, 236, (1978), 275~285.
4. P. Eakin, *On Arnolds formula for the dimension of a polynomial ring,* Proc. of A. M. S, 54, (1976), 11~15.
5. S. Goto and K. Watanabe, *On Graded rings* 1, J. of Math. Soc. Japan, 30, (1978), 179~213.
6. C. Huneke, *Powers of ideal generated by Weak d-sequences,* J. of Alg, 68, (1981), 471~509.
7. S. Mac Lane, *Homology,* Springer-Verlag, (1975).
8. H. Matsumura, *Commutative Algebra,* W. A. Benjamin. Inc, (1980).
9. K. McDowell, *A codimension theoem for Pseudo-Noetherian rings,* Trans. of A. M. S, 214, (1975), 179~185.
10. M. Nagata, *Local rings,* Interscience Pub., (1962).
11. J. J. Rotman, *An introduction to Homological Algebra,* Academic Press Inc., (1979).
12. P. Samuel and O. Zariski, *Commutative Algebra Vol.* 1 *and* 2, Van Nostrand Co. Inc., (1958).
13. A. Simis and W. V. Vasconcelos, *The Syzygies of the conormal module,* Amer. J. Math, 103, (1981), 203~224.
14. S. Raghavan Balwant Singh R. Sridharan, *Homological methods in Commutative Algebra,* Oxford University Press, (1975).
15. K. Yoshid, *On flat over-rings of a Krull domain,* J. of Math. Soc. Japan, 54, (1976), 11~15.

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