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Einstein generic submanifolds of an even-dimensional Euclidean space

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§1. Introduction

Let E^{2m} be a 2m-dimensional Euclidean space and O the origin of a Cartesian coordinate system in E^{2m}, and denote by $\bar{X}$ the position vector representing a point of E^{2m} with respect to the origin. Since E^{2m} is even-dimensional, E^{2m} can be regarded as a flat Hermitian manifold, and hence there exists a tensor field F of type (1, 1) with constant components such that

(1.1) $$F^2=-I,\quad (F\bar{X})\cdot(F\bar{Y})=\bar{X}\cdot\bar{Y}$$

for any vectors $\bar{X}$ and $\bar{Y}$, where I denotes the identity transformation, a dot the inner product in the Euclidean space.

A submanifold M of a Euclidean space E^{2m} is called a generic (an ***anti-holomorphic***) submanifold if the normal space $T_P^{\perp}(M)$ of M at any point $P\in M$ is always mapped into the tangent space $T_P(M)$ under the action of the almost complex structure tensor F of the ambient space E^{2m}, that is, $FT_P^{\perp}(M)\subset T_P(M)$ for all $P\in M$ (see [2], [3], [5] and [9]).

The f-structure induced on the generic submanifold M of E^{2m} is said to be ***normal*** if the second fundamental tensors of M and the f-structure commute. ([2], [4]).

On the other hand, Pak and one of the present authors [2] studied generic submanifolds of an even-dimensional Euclidean space and proved the following:

Theorem A. *Let M be an n-dimensional complete generic submanifold with flat normal connection of a 2 m-dimensional Euclidean space E^{2m}. If the f-structure induced on M is normal and the mean curvature vector is parallel in the normal bundle, then M is a sphere $S^n(r)$ of dimension n, an n-dimensional plane E^n $(\subset E^{2m})$, a pythagorean product of the from*

(1.2) $S^{p_1}(r_1)\times\cdots\times S^{p_N}(r_N)$, $p_1,\cdots,p_N\geqq 1$, $p_1+\cdots+p_N=n$, $1<N\leqq 2m-n$,

(1.3) $S^{p_1}(r_1)\times\cdots\times S^{p_N}(r_N)\times E^p$, $p_1,\cdots,p_N$, $p\geqq 1$, $p_1+\cdots+p_N+p=n$,

$1<N\leqq 2m-n$, where $S^p(r)$ is a p-dimensional sphere with radius $r>0$, E^p

a p-dimensional plane. If M is a pythagorean product of the form (1.2) *or* (1.3), *then M is of essential codimension N.*

The purpose of the present paper is to explore some characterizations of Einstein generic submanifolds with flat normal connection of an even-dimensional Euclidean space. Our main results will be proved in § 3.

§ 2. Preliminaries

Let M be an n-dimensional Riemannian manifold immersed isometrically in an even-dimensional Euclidean space E^{2m} by the immersion $i: M \to E^{2m}$.

We denote by B the differential of i, that is,

$$B = di : T_P(M) \to T_{i(P)}(E^{2m})$$

for each point $P \in M$. A Riemannian metric is induced on M from that of E^{2m} in such a way that

$$g(X, Y) = BX \cdot BY \tag{2.1}$$

for any vectors X and Y in M.

We denote by N_A $2m-n$ mutually orthogonal unit normals to M. Here, in the sequel, the indices $A, B, C, \cdots$ run over the range $\{n+1, \cdots, 2m\}$.

Throughout this paper we assume that the generic submanifold M is immersed in an even-dimensional Euclidean space E^{2m} and W, X, Y and Z are vector fields in M. Then, by the definition of generic submanifolds, we can put in each coordinate neighborhood

$$FBX = BfX - \sum_A u_A(X) N_A \tag{2.2}$$

$$FN_A = BU_A, \tag{2.3}$$

where f is a tensor field of type (1, 1) defined on M, u_A a 1-form and U_A a vector field associated with u_A given by $g(U_A, X) = u_A(X)$.

Operating F to (2.2) and (2.3) respectively, and using (1.1) and those equations, we can easily find [2]

$$\left\{\begin{aligned} & f^2 = -I + \sum_A u_A \otimes U_A, \\ & fU_A = 0, \quad u_A \circ f = 0, \\ & u_A(U_B) = \delta_{AB}, \\ & g(fX, fY) = g(X, Y) - \sum_A u_A(X)\, u_A(Y), \end{aligned}\right. \tag{2.4}$$

which implies $f^3 + f = 0$. Consequently M admits the so-called f-structure satisfying $f^3 + f = 0$ (see [6] and [8]).

If we put

$$\tilde{f}(X, Y)=g(fX, Y), \tag{2.5}$$

then we can easily verify that

$$\tilde{f}(X, Y)=-\tilde{f}(Y, X).$$

The operator of covariant differentiation with respect to the Riemannian connection in E^{2m} (resp. M) will be denoted by $\bar{\nabla}$ (resp. ∇). Then the Gauss and Weingarten formulas are respectively given by

$$\begin{cases} \bar{\nabla}_{BX} BY=B\nabla_X Y+h(X, Y), \\ \bar{\nabla}_{BX} N_A = -BH_A X + D_X N_A, \end{cases} \tag{2.6}$$

where D denotes the covariant differentiation with respect to the linear connection induced in the normal bundle of M. h and H are both called the second fundamental tensors of M and are related by

$$h(X, Y)\cdot N_A= g(H_A X, Y)= h_A(X, Y), \tag{2.7}$$

where h_A denotes the second fundamental tensors associated with the normal vector fields N_A, that is,

$$h(X, Y)= \sum_A h_A(X, Y) N_A.$$

For the second fundamental tensor h we define its covariant derivative $\nabla_X h$ by

$$(\nabla_X h)(Y, Z)=D_X(h(Y, Z))-\sum_A \{ h_A(\nabla_X Y, Z)+h_A(Y, \nabla_X Z)\} N_A. \tag{2.8}$$

Now, differentiating (2.2) and (2.3) covariantly along M respectively and using (2.6) and (2.8), we find [2]

$$(\nabla_Y f)X= \sum_A h_A(Y, X) U_A-\sum_A u_A(X) H_A Y, \tag{2.9}$$

$$(\nabla_Y u_A)X=h_A(Y, fX), \quad \nabla_Y U_A=-fH_A Y, \tag{2.10}$$

$$u_B(H_A Y)= u_A(H_B Y). \tag{2.11}$$

We now define a tensor field S of type (1, 2) given by

$$S=N+\sum_A du_A\otimes U_A, \tag{2.12}$$

where N is the Nijenhuis tensor formed with the f-structure f, that is,

$$N(Y, X)=[fY, fX]-f[Y, fX]-f[fY, X]+f^2[Y, X].$$

When the tensor field S vanishes identically, the f-structure induced on M is said to be ***normal*** (see [2] and [4]).

But, for the generic submanifold M of the Euclidean space E^{2m}, taking account of (2.9) and (2.10), (2.12) can be written of the form :

$$S(Y, X)=\sum_A \{ (H_A fY-fH_A Y)u_A(X)-(H_A fX-fH_A X)u_A(Y) \}.$$

Thus, we have

Lemma 2. 1 ([2]). *Let M be an n-dimensional generic submanifold of E. Then the f-structure induced on M is normal if and only if*

(2.13) $$H_A f = f H_A.$$

From this, it follows that

(2.14) $$h_A(fX,\ Y) + h_A(X,\ fY) = 0.$$

In fact, using (2.13), we get

$$g(H_A fX,\ Y) = g(fH_A X,\ Y),$$

or, using (2.5) and (2.7),

$$h_A(fX,\ Y) = \bar{f}(H_A X,\ Y) = -\bar{f}(Y,\ H_A X) = -g(fY,\ H_A X) = -h_A(X,\ fY).$$

Since the ambient manifold is Euclidean, equations of Gauss, Codazzi and Ricci for M are respectively given by

(2.15) $$K(X,\ Y)Z = \sum_A \{H_A X h_A(Y,\ Z) - h_A(X,\ Z)H_A Y\},$$

(2.16) $$(\nabla_Y h_A)(Z,\ X) - (\nabla_Z h_A)(Y,\ X) = 0.$$

(2.17) $$K^N(X,\ Y)N_A = \sum_B g([H_A,\ H_B]X,\ Y)N_B,$$

where K and K^N are the curvature tensor of M and that of the connection in the normal bundle, and we have put $[H_A,\ H_B] = H_A H_B - H_B H_A$.

Applying the f-structure f to (2. 13) and using its commutativity, we find

$$H_A f^2 X = f^2 H_A X,$$

for any vector fixeld X in M, or, using (2. 4).

$$\sum_B u_B(X) H_A U_B = \sum_B u_B(H_A X) U_B.$$

Putting $X = U_C$ in the above equation and making use of (2. 4), we get

(2.18) $$H_A U_C = \sum_B P_{BAC} U_B,$$

where we have put

(2.19) $$P_{BAC} = u_B(H_A U_C).$$

But, using $g(U_A, X) = u_A(X)$, we find

$$u_B(H_A U_C) = g(H_A U_C,\ U_B) = g(U_C,\ H_A U_B) = u_C(H_A U_B)$$

because H_A is the symmetric operator, that is, P_{BAC} is symmetric for the indices B and C. Since we can see from (2. 11) that P_{BAC} is symmetric for the indices B and A, P_{BAC} is symmetric for all indices.

If we apply the second fundamental tensor H_D to (2. 18) and use itself, then we have

(2.20) $$H_D H_A U_C = \sum_{B,E} P_{BAC} P_{EDB} U_E.$$

Assuming the conndction of the normal bundle of M is flat, that is, $K^N(X, Y)N_B = O$ for each vector X and Y, or, equivalently $H_A H_B = H_B H_A$, (2.20) implies that

(2.21) $$\sum_B P_{BAC} P_{EDB} = \sum_B P_{BDC} P_{EAB},$$

for which, we find

(2.22) $$\sum_B P_B P_{EDB} = \sum_{B,A} P_{BAD} P_{EAB},$$

where we have put

$$P_B = \sum_A P_{BAA}.$$

We now prepare

Lemma 2.2 ([2]). ***Let M be a generic submanifold with flat normal connection of an even-dimensional Eucldean space. If the f-structure induced on M satisfies* (2.13), *then we have***

(2.23) $$H_B H_A = \sum_C P_{CBA} H_C,$$

(2.24) $$\nabla_X TrH_A = \nabla_X P_A,$$

where TrH_A denotes the trace of H_A.

Proof) Since (2.18) implies

$$h_A(U_C, X) = \sum_B P_{BAC}\, g(U_B, X),$$

for any vector field X, differentiating this covariantly, we find

$$(\nabla_Y h_A)(U_C, X) + h_A(-fH_C Y, X) = \sum_B P_{BAC} g(-fH_B Y, X) + \sum_B (\nabla_Y P_{BAC})\, g(U_B, X)$$

with the help of (2.10). Taking account of (2.16), the equations of Codazzi, we get

$$h_A(fH_C X, Y) - h_A(fH_C Y, X) = \sum_B P_{BAC} \{ g(-fH_B Y, X) + g(fH_B X, Y) \} + \sum_B \{ (\nabla_Y P_{BAC}) g(U_B, X) - (\nabla_X P_{BAC})\, g(U_B, Y) \},$$

or, using (2.13) and (2.14),

(2.25) $$2h_A(fH_C X, Y) = 2\sum_B P_{BAC}\, g(fH_B X, Y) + \sum_B \{ (\nabla_Y P_{BAC}) g(U_B, X) - (\nabla_X P_{BAC})\, g(U_B, Y) \}.$$

Putting $X = U_D$ in this equation and taking account of (2.4) and (2.13), we find

(2.26) $$\nabla_Y P_{DAC} = \sum_B (\nabla_{U_D} P_{BAC})\, g(U_B, Y).$$

If we substitute this into (2.25), then we obtain

$$h_A(fH_C X, Y) = \sum_B P_{BAC}\, g(fH_B X, Y),$$

or, equivalently

$$g(fH_A H_C X,\ Y)=\sum_B P_{BAC}\ g(fH_B X,\ Y)$$

because of (2.13). Consequently, we have

$$fH_A H_C X=\sum_B P_{BAC} fH_B X.$$

Applying the f-structure f to this and using (2.4), we get

$$(2.27)\qquad -H_A H_C X+\sum_B u_B(H_A H_C X)U_B=-\sum_B P_{BAC} H_B X+\sum_{B,E} P_{BAC} u_E(H_B X)U_E.$$

Since $\sum_B u_B(H_A H_C X)U$ can be written as

$$\sum_B u_B(H_A H_C X)U_B=\sum_B g(H_A H_C X,\ U_B)U_B=\sum_B g(X,\ H_C H_A H_B)U_B=\sum_{B,D,E} P_{ECD} P_{DAB}\ g(X,$$

$U_E)U_B$ because of (2.7) and (2.18), and $\sum_B\sum_E P_{BAC}\, u_E(H_B X)\, U_E$ as

$$\sum_{B,E} P_{BAC}\ g(H_B X,\ U_E)U_E=\sum_{B,E} P_{BAC}\ g(X,\ H_B U_E)U_E=\sum_{B,D,E} P_{BAC} P_{DBE}\ g(X,\ U_D)U_E,$$

(2.27) reduces to

$$H_A H_C X=\sum_B P_{BAC} H_B X$$

because of (2.21), which shows that (2.23) is proved. For the proof of (2.24), see Lemma 2.2 of [2].

§ 3. Some characterizations of Einstein generic submanifolds of an even-dimensional Euclidean space

We first prove

Lemma 3.1. *Under the same assumptions as those stated in Lemma* 2.2, *we have*

$$(3.1)\qquad \sum_{A,B}(\nabla_Y P_A)P_{CBA}(TrH_B-P_B)=0$$

for any vector field Y.

Proof) From (2.26), we have

$$(3.2)\qquad \nabla_Y P_A=\sum_B(\nabla_{U_A} P_B)u_B(Y).$$

which and (2.4) imply that

$$(3.3)\qquad \nabla_{f^2 Y} P_A=0.$$

Differentiating (3.2) covariantly and taking account of (2.10), we find

$$\nabla_X \nabla_Y P_A=\sum_B[(\nabla_X \nabla_{U_A} P_B)u_B(Y)+(\nabla_{U_A} P_B)\{h_B(X,\ fY)+u_B(\nabla_X Y)\}],$$

from which, using the Ricci identity,

$$(3.4)\qquad \sum_B\{(\nabla_X \nabla_U P_B)u_B(Y)-(\nabla_Y \nabla_{U_A} P_B)u_B(X)+2(\nabla_{U_A} P_B)h_B(X,\ fY)\}=0$$

because of (2.14) and $K^N(X,\ Y)=0$. Setting $Y=U_C$, it follows that

$$\nabla_X \nabla_{U_A} P_C=\sum_B(\nabla_{U_C}\nabla_{U_A} P_B)u_B(X),$$

from which, we get

$$\nabla_{U_D} \nabla_{U_A} P_C = \nabla_{U_C} \nabla_{U_A} P_D$$

because of (2.4). Thus, (3.4) reduces to

$$\sum_B (\nabla_{U_A} P_B) h_B(X, fY) = 0.$$

Which implies

$$\sum_{B,A} (\nabla u_A(Z)_{U_A} P_B) h_B(X, fY) = 0,$$

or, using (2.4) and (3.3), we have

$$(3.5) \qquad \sum_B (\nabla_Z P_B)\, h_B(X, fY) = 0$$

for arbitrary vector fields X, Y and Z.

Putting $X = fW$ and using (2.4), we obtain

$$\sum_B (\nabla_Z P_B) \{ h_B(W, Y) - \sum_A u_A(Y) h_B(W, U_A) \} = 0$$

with the aid of (2.4), or, equivalently

$$(3.6) \qquad \sum_B (\nabla_Z P_B) \{ g(H_B W, Y) - \sum_A u_A(Y) g(H_B W, U_A) \} = 0.$$

Let $\{E_1, E_2, \cdots, E_n\}$ be the set of orthonormal bases of tangent space $T_P(M)$ for each point $P \in M$. Then, (3.6) implies that

$$\sum_i \sum_B (\nabla_Z P_B) \{ g(H_B H_C E_i, E_i) - \sum_A u_A(E_i) g(H_B H_C E_i, U_A) \} = 0,$$

which means

$$(3.7) \qquad \sum_i \sum_B (\nabla_Z P_B) \{ \sum_D P_{DBC}\, g(H_D E_i, E_i) - \sum_{A,D,E} P_{DBC} P_{EDA}\, u_A(E_i) u_E(E_i) \} = 0$$

because of (2.18), (2.23) and H_A being the symmetric operator.

Since $u_A(E_i) = g(U_A, E_i)$, we can see that

$$u_A(E_i) = U_{Ai},$$

where we have put $U_A = \sum_A U_{Ai} E_i$. But, $g(U_A, U_B) = \delta_{AB}$ implies $\sum_i U_{Ai} U_{Bi} = \delta_{AB}$.

Therefore, (3.7) reduces to

$$\sum_{B,D} (\nabla_Z P_B)\, P_{DBC} \{ \sum_i g(H_D E_i, E_i) - P_D \} = 0$$

because P_{ABC} is symmetric for all indices. Thus Lemma 3.1 is proved.

If $E_1, E_2, \cdots, E_n$ are orthonormal vector fields, then we get the Ricci tensor given by

$$R(Y, Z) = \sum_i g(K(E_i, Y) Z, E_i),$$

which implies the scalar curvature

$$R^* = \sum_i R(E_i, E_i).$$

Taking account of (2.15) and these facts, we have the Ricci tensor and the scalar curvaure respectively of the form :

(3.8) $$R(Y, Z)=\sum_A \{ TrH_A h_A(Y, Z)-\sum_i h_A(E_i, Z)h_A(E_i, Y) \},$$

(3.9) $$R^*=\sum_A \{ (TrH_A)^2 - Tr\ (H_A^2) \}.$$

A submanifold M of E^{2m} is called *proper Einstein* if it satisfies

(3.10) $$R(Y, Z)=R^*/n\ g(Y, Z),\ R^* \neq 0.$$

We now prove

Theorem 3.2. *Let M be an n-dimensional complete proper Einstein submanifold with flat normal connection of a $2m$-dimensional Euclidean space E^{2m}. If the f-structure induced on M is normal, then M is a sphere $S^n(r)$ of dimension n, a pythagorean product of the form*

(*) $S^{p_1}(r)\times\cdots\times S^{p_N}(r)$, $p_1, \cdots p_n$ are odd numbers $\geqq 1$, $p_1=\cdots=p_N$, $Np=n$, $1<N\leqq 2m-n$, where $S^p(r)$ is p-dimensional sphere with radius $r>0$. If M is a pythagorean product of the form (*), then M is of essential codimension N.

Proof) Since the f-structure induced on M is normal, we see from Lemma 2.1 that Lemma 2.2 and Lemma 3.1 are valid. Thus, it follows that

$$R(Y, Z)=\sum_A (TrH_A-P_A)h_A(Y, Z)$$

because of (2.7), (2.23) and (3,8), which implies

(3.11) $$\sum_A (TrH_A-P_A)h_A(Y, Z)=R^*/n\ \ g(Y, Z),\ R^* \neq 0$$

since the submanifold is proper Einstein.

Letting $Y=U_B$ and $Z=U_C$, and taking account of (2.4), (2.7) and (2.18), we obtain

$$\sum_A (TrH_A-P_A)P_{CBA}=R^*/n\ \delta_{CB}$$

because P_{ACB} is symmetric for any index.

This together with Lemma 3.1 gives $\nabla_X P_A=0$ for any vector field X since $R^* \neq 0$ and hence $\nabla_X TrH_A=0$ because of (2.24), that is, the mean curvature vector is parallel in the normal bundle. According to Theorem A and this fact, we have the conclusions of the theorem since M is Einstein.

Lemma 3.3. *Let M be an n-dimensional generic submanifold with flat normal connection of a $2m$-dimensional Euclidean space E^{2m}. If the f-structure induced on M is normal, then the scalar curvature R^* of M is zero if and only if M is locally Euclidean.*

Proof) Taking account of (2.23), (3.9) becomes

$$R^*=\sum_A (TrH_A-P_A)TrH_A. \tag{3.12}$$

We now compute the square of the length of $h_A-\sum_{B,C}P_{BAC}u_B\otimes u_C$. If E_1, E_2, $\cdots$, E_n are local orthonormal vector fields, then we get

$$\| h_A-\sum_{B,C}P_{BCA}u_B\otimes u_C\|^2=\sum_A\sum_{i,j}\{h_A(E_i,E_j)-\sum_{B,C}P_{BCA}u_B(E_i)u_C(E_j)\}\{h_A(E_i,E_j)-\sum_{D,E}P_{DEA}u_D(E_i)u_E(E_j)\}$$

$$=\sum_A\sum_{i,j}\{h_A(E_i,E_j)h_A(E_i,E_j)-2\sum_{B,C}P_{BCA}u_B(E_i)u_C(E_j)h_A(E_i,E_j)+\sum_{B,C,D,E}P_{BCA}P_{DEA}u_B(E_i)u_C(E_j)u_D(E_i)u_E(E_j)\}.$$

Since we have already shown in the proof of Lemma 3.1 that $\sum_i u_B(E_i)u_C(E_i)=\delta_{BC}$ and $U_B=\sum_i U_{Bi}E_i$, the above equation can be deformed as follows:

$$\| h_A-\sum_{B,C}P_{BCA}u_B\otimes u_C\|^2=\sum_A\sum_i g(H_AH_AE_i,E_i)-2\sum_{B,C}\sum_{i,j}P_{BCA}\,g(H_AE_i,E_j)U_{Bi}U_{Cj}+\sum_{B,C,A}P_{BCA}P_{BCA} \tag{3.13}$$

$$=\sum_A P_ATrH_A-2\sum_{B,C}P_{BCA}\,g(H_AU_B,U_C)+\sum_{B,C,A}P_{BCA}P_{BCA}$$

$$=\sum_A P_ATrH_A-\sum_{B,C,A}P_{BCA}P_{BCA}=\sum_A(TrH_A-P_A)P_A$$

with the help of (2.18), (2.21), (2.23) and the symmetric character of P_{BCA}. Therefore, we see from (3.12) and (3.13) that

$$R^*=\|TrH_A-P_A\|^2+\|h_A-\sum_{B,C}P_{BCA}\,u_B\otimes u_C\|^2.$$

If the scalar curvature R^* is zero, then we have

$$TrH_A=P_A \quad\text{and}\quad h_A=\sum_{B,C}P_{BCA}\,u_B\otimes u_C. \tag{3.14}$$

Substituting the second relationship of (3.14) into (2.15), we get

$$g(K(X,Y)Z,W)=\sum_A\{h_A(X,W)h_A(Y,Z)-h_A(X,Z)h_A(Y,W)\}$$
$$=\sum_{A,B,C,D,E}(P_{BCA}P_{DEA}-P_{BEA}P_{DCA})u_B(X)u_C(W)u_D(Y)u_E(Z),$$

or, using (2.21),

$$g(K(X,Y)Z,W)=0,$$

that is, $K(X,Y)Z=0$. Consequently, M is locally Euclidean. Conversely, if M is locally Euclidean, then the scalar curvature R^* of M is clearly zero.

Finally we prove

Theorem 3. 4. *Let M be an n-dimensional complete locally irreducible ge-*

neric submanifold with flat normal connection of a $2m$*-dimensional Euclidean space* E^{2m}*. If the* f*-structure induced on* M *is normal and the square of the length of the Ricci tensor is constant, then* M *is of the same type as those stated in Theorem* 3.2.

Proof) From our assumptions (2.1)~(3.9) are valid. Setting $W=E_i$ and $Y=E_i$ in (3.6), and summing with respect to i ($i=1,2,\cdots,n$), we have

$$\sum_B (\nabla_Z P_B)(TrH_B - P_B) = 0,$$

where $\{E_1, E_2, \cdots, E_n\}$ is the set of orthonormal vector fields, from which, taking account of (2.24),

$$\sum_A (\nabla_Z TrH_A)(TrH_A - P_A) = 0. \tag{3.15}$$

If we use (2.7) and (2.23), then (3.8) and (3.9) becomes respectively

$$R(Y, Z) = \sum_A (TrH_A - P_A) h_A(Y, Z) \tag{3.16}$$

and

$$R^* = \sum_A TrH_A (TrH_A - P_A). \tag{3.17}$$

Differentiating (3.17) covariantly and using (2.24) and (3.15), we see that the scalar curvature R^* is constant on M.

From the Ricci identity and the fact that the normal connection is flat, the following identity is induced:

$$(\nabla_X \nabla_Y \nabla h_A)(Z, W) - (\nabla_Y \nabla_X h_A)(Z, W) = (\nabla_{(X,Y)} h_A)(Z, W) - h_A(K(X, Y)Z, W) - h_A(Z, K(X, Y)W),$$

which implies

$$\sum_i (\nabla_{E_i} \nabla_{E_i} h_A)(X, Y) - \nabla_X \nabla_Y TrH_A = R(X, H_A Y) - \sum_i h_A(K(E_i, Y)X, E_i)$$

with the aid of (2.16). Substituting (2.15) and (3.16) into the right hand side of this, and using (2.22) and (2.23), we obtain

$$\sum_i (\nabla_{E_i} \nabla_{E_i} h_A)(X, Y) - \nabla_X \nabla_Y TrH = 0. \tag{3.18}$$

Operating $\sum_i \nabla_{E_i} \nabla_{E_i}$ to (3.16) and making use of (2.24), we find

$$\sum_i (\nabla_{E_i} \nabla_{E_i} R)(X, Y) = \sum_A (TrH_A - P_A) \nabla_X \nabla_Y TrH_A,$$

which becomes

$$\sum_i (\nabla_{E_i} \nabla_{E_i} R)(X, Y) = \nabla_X \nabla_Y R^*$$

because of (2.24) and (3.17). Since the scalar curvature R^* is constant on M, it follows that

$$(3.19)\qquad \sum_i (\nabla_{E_i}\nabla_{E_i}R)(X, Y)=0 .$$

Therefore, the identity :

$$(3.20)\quad \frac{1}{2}\triangle \| R \|^2 = \sum_{i,j,k} \{(\nabla_{E_i}\nabla_{E_i}R)(E_j, F_k)\} R(E_j, E_k) + \| \nabla R \|^2$$

gives $\nabla R = 0$ because of (3.19) and the assumption that $\| R \|^2$ is constant, where $\triangle$ is the Laplacian given by $\triangle = \sum_i \nabla_{E_i} \nabla_{E_i}$. Hence we have

$$R(X, Y)=R^* / n\ g(X, Y)$$

since M is locally irreducible. But, Lemma 3.3 shows that the scalar curvature R^* cannot be zero. Thus, the submanifold M is proper Einstein. After all M is of the same type as those stated in Theorem 3.2.

Replacing the condition $\| R \|^2$=constant in Theorem 3.4 by the compactness, we have from (3.19) and (3.20) that the Ricci tensor is parallel. Thus, according to Theorem 3.4, we have

Corollary 3. 5. *Let M be an n-dimensional compact locally irreducible generic submanifold with flat normal connection of a 2m-dimensional Eucldean space E^{2m}. If the f-structure induced on M is normal, then M is a sphere $S^n(r)$ of dimension n or a pythagorean product of the form*

$S^{p_1}(r) \times \cdots \times S^{p_N}(r)$, $p_1, \cdots, p_N$ are odd numbers $\geqq 1$, $p_1 = \cdots = p_N$, $Np=n$,

$1 < N \leqq 2m-n$,

where $S^p(r)$ is a p-dimensional sphere with radius $r>0$. If M is a pythagorean product of the form above, then M is of essential codimension N.

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