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A note on Thom isomorphism in k-Theory

By Yong Woon Kim, Kee An Lee

Throughout this paper we assume that every vector bundle is a C-vector bundle (each fiber is a C-vector space, C is the field of complexes). Let G be a finite abelian group. For a left G-space X a ***G-vector bundle*** over X is given a vector bundle E satisfying the following conditions :

(i) E is a left G-space with $\theta : G\times E \to E$.

(ii) the diagram

$$\begin{array}{ccc} G\times E & \xrightarrow{\theta} & E \\ {\scriptstyle 1_G\times\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ G\times X & \longrightarrow & X \end{array}$$

is commutative,

(iii) $E_x \to E_{gx}$ $(e \mapsto \theta(ge))$ is a C-linear for

$x\in X$ and $g\in G$.

The purpose of this paper is to prove the Thom isomorphism theorem with respect to complex k-theory of G-bundle (Theorem 5 and 6).

For a topological space Y, $\mathcal{E}(Y)$ denotes the category consisting of all vector bundles over Y and vector bundle morphisms. Also, when X is a G-space $\mathcal{E}_G(X)$ is the catgory of G-vector bundle over X. $K_C(Y)=K(Y)$ is the completion of the semi-group $\mathcal{E}(Y)$, and $K_G(X)$ is the completion of the semi-group $\mathcal{E}_G(X)$.

Lemma 1. For a finite group G, let X be a left G-space, on which G acts freely. For a G-vector bundle E over X, E/G is a vector bundle over X/G and $\pi^*(E/G)\approx E$, where $\pi : X\to X/G$ is the canonical projection. Moreover the categories $\mathcal{E}_G(X)$ and $\mathcal{E}(X/G)$ are equivalent under π^*.

Proof. In the above condition (iii)

$$E_x \longrightarrow E_{gx} \quad (e \longmapsto \theta(g, e))$$

is a linear isomorphism, because of that

$$E_{gx} \longrightarrow E_x \quad (f \mapsto \theta(g^{-1}, f))$$

implies that for each e E_x $\theta(g^{-1}, \theta(g, e)) = \theta(g^{-1} g, e) = e$. we introduce the equivalence relation "$\sim$" on E as follows.

For $v, w \in E$

$$v \sim w \leftrightarrow \exists g \in G \quad \cdot \ni \cdot \quad w = gV.$$

Then, it is clear that $E/\sim = E/G$.

Furthermore, X/G is the set $\{ G_x \mid x \in X \}$ with the quotient topology. Therefore, if we define $\widetilde{\pi}: E/G \rightarrow X/G$ by $[E_x] \mapsto G_x$ then (E/G) $Gx \approx Ex$, where $[E_x]$ is an element of $E/\sim = E/G$ and E_x is the fiber of E at $x \in X$.

Let Ux be an open neighborhood of $x \in X$ such that $E \mid U_x$ is trivial. Then, for all $g \in G$ g U_x is an open neighborhood of g_x such that $U_x \approx g$ U_x (homeomorphic). Therefore $\pi(U_x) = \{ GY \mid Y \in U_x \}$ which is an open neighborhood of G_x in X/G.

Define $\psi: \widetilde{\pi}^{-1}(\pi(U_x)) \rightarrow \pi(U_x) \times C$ by

$$\varphi([E_x]) = G_x \times \varphi(E_x),$$

where $\varphi: \bar{\pi}^{-1}(U_x) \rightarrow U_x \times C^n$ is a homeomorphism for the trivialization domain U_x and $\bar{\pi}: E \rightarrow X$. Hence $\widetilde{\pi}: E/G \rightarrow X/G$ is locally trivial, and thus E/G is a vector bundle over X/G.

By the above description it is clear that $E \approx \pi^*(E/G)$. Next, we define a functor $s : \xi_G(x) \rightarrow \xi(X/G)$ by $S(E) = E/G$. Then we have $\pi^* \circ S \approx 1_{\xi_G}(X)$ and $S \cdot \pi^* \approx 1_{\xi(X/G)}$. *Q. E. D.*

The following corollary is obvious by Lemma 1.

Corollary 2. and $K(X/G)$ are isomorphic as abelian groups.

Definition 3. Let Y be a locally compact space. We consider the full subcategory $\xi'(Y)$ of $\xi(Y)$ whose objects are the direct summands of trivial bundles. For $E \in \xi'(Y)$ if E has a metric and if $E_0 \oplus E_1$ is an orthogonal decomposition of E with respect to this metric, then an endomorphism $D: E \rightarrow E$ is said to be ***admissible*** if

(a)
$$D = \begin{pmatrix} O & \alpha^* \\ \alpha & O \end{pmatrix}.$$

where $\alpha: E_0 \rightarrow E_1$ and α^* is the adjoint of α,

(b) there is a compact subset K of Y such that $D \mid Y-K$ is an automorphism of $E \mid Y-K$.

We put

$$\xi=\{(\mathrm{E}, \mathrm{D}) \mid E\in\xi'(Y) \text{ and } D \text{ is admissible}\}$$

An element $(E, D)\in\xi$ is called ***elementary*** if D is an automorphism. For two elements (E, D) and (E', D') are ***homotopic*** if there exists an isometry of the form $f_0 \oplus f_1$, where $f_i : E_i \to E_i'$ $(i=0, 1)$, such that $f^{-1}\circ D'\circ f$ is homotopic within the admissible operators on E.

We introduce the equivalence relation "$\sim$" on ξ as follows. For σ, $\sigma'\in\xi$

$$\sigma\sim\sigma' \leftrightarrow \exists\ \tau,\ \tau' \text{ elementary, such that}$$

$$\sigma+\tau \text{ is homotopic to } \sigma'+\tau'.$$

where for (E, D), $(E', D')\in\xi$

$$(E, D)+(E', D')=(E\oplus E', D\oplus D').$$

We put $$K_0(Y)=\xi/\sim.$$

in which each element is denoted by $\sigma(E, D)$. Then we can prove that $K_0(Y)$ and $K(Y)$ are isomorphic ([3]).

Let Y be compact, and let $\pi : V\to Y$ be a vector bundle over Y with metric $\langle,\rangle=\{\langle,\rangle_y \mid y\in Y\}$. Then, we have the exterior bundle $\wedge V$ over Y such that for $y\in Y$ $(\wedge V)_y=\wedge V_y$ which is the exterior algebra of V_y. There is the metric of $\wedge V$ as follows ([1]) :

(c) if $j\neq k$ then for $y\in Y$ $\wedge^j V_y$ is orthogonal to $\wedge^k V_y$.

(d) if $x=u_1\wedge --- \wedge u_k$, and $Y=V_1\wedge --- \wedge V_k$, where u_r, $V_s\in Vy$, then

$$\langle x, y\rangle_y=\det \mid \langle u_r, V_s\rangle_y \mid$$

For a vector $v\in V_y$, we let $d_v : \wedge V_y \to \wedge V_y$ denote the linear map defined by $d_V(e)=V\wedge e$. Let us denote the adjoint of d_v by ∂_v. The Thom class $U_v\in k(V)$ $(=k_c(V))$ may be described as follows ([3]).

$$U_v=\sigma(\pi^*(\wedge V), \triangle),$$

where $\triangle : \pi^*(\wedge V)\to\pi^*(\wedge V)$ is defined by $\triangle(y, v)=d_v+\partial_v$ at the point (y, v) for $y\in Y$ and $v\in V_y$. Note that

(e) $\triangle$ is admissible

(f) $\pi^*(\wedge V)=\pi^*(\wedge^{(0)}V)\oplus\pi^*(\wedge^{(1)}V)$ where

$$\wedge^{(0)}V=\bigoplus_{i=0}^{\infty}\wedge^{2i}V \qquad \wedge^{(1)}V=\bigoplus_{i=0}^{\infty}\wedge^{2i+1}V$$

We put

$$k^*(V)=\bigoplus_{q=0}^{\infty}k^{-q}(V),\quad k^*(Y)=\bigoplus_{q=0}^{\infty}k^{-q}(Y).$$

Then we have the Thom isomorphism theorem such that

Theorem 4. $k^*(V)$ is a free $k^*(Y)$-module of rank one, generated by the Thom class U_v (for proof see [3]).

We shall return to our theory. As before, we assume that

(iv) G is a finite abelian group with discrete topology,

(v) x is a left G-space and compact,

(vi) G acts freely on X,

(vii) $\widetilde{\pi}: V\rightarrow X$ is a G-vector bundle over x and $\pi: X\rightarrow X/G$ is the canonical projection.

Theorem 5. $K_G(V)$ is a free $K_G(X)$-module of rank one, generated by the Thom class $\widetilde{\pi}^*(U_{V/G})=U_V$ where $\widetilde{\pi}: V\rightarrow V/G$ is the canonical projection.

Proof. We have to note that there are isomorphisms

$$\pi^*: K(X/G)\xrightarrow{\approx} K_G(X),\quad \widetilde{\pi}^*: K(V/G)\xrightarrow{\approx} K_G(V)$$

by corollary 2, where π^* and $\widetilde{\pi}^*$ are induced from

$$\pi^*: \xi(X/G)\longrightarrow \xi_G(X) \text{ and } \widetilde{\pi}^*: \xi(V/G)\longrightarrow \xi_G(V),$$

respectively. Moreover, we know that $K(V/G)$ is a free $K(X/G)$-module of rank one, generated by $U_{V/G}=\sigma(\widetilde{\pi}^*(\wedge(V/G)),\ \triangle)$ by Theorem 4.

Since $X\times V$ is a left G-space with $G(X\times V)=GX\times GV$, there is the product

$$K_G(X)\times K_G(V)\xrightarrow{\cup} K_G(X\times V)\xrightarrow{j^*} K_G(V),$$

wher $\cup$ is the cup-product in k-theory ([2]) and $j: V\rightarrow X\times V$ is defined by $j(V_x)=(x,\ V_x)$ for $x\in X$ and $V_x\in V_x$, Since $X\times V/G\approx X/G\times V/G$ and G acts freely on $X\times V$ we have the commutative diagram

$$\begin{array}{ccccc}
K(X/G)\times K(V/G) & \xrightarrow{\cup} & K(X/G\times V/G) & \xrightarrow{j^*} & K(V/G)\\
\pi^*\times\widetilde{\pi}^* \downarrow\approx & & \downarrow\approx & & \approx\downarrow\widetilde{\pi}^*\\
K_G(X)\times K_G(V) & \xrightarrow{\cup} & K_G(X\times V) & \xrightarrow{j^*} & K_G(V).
\end{array}$$

By Theorem 4 we have

$$K(X/G)\times U_{V/G}\xrightarrow{\approx} K(V/G),$$

and thus

$$K_G(X)\times\widetilde{\pi}^*(U_{V/G})=K_G\times U_V\xrightarrow{\approx}K_G(V).\qquad Q.\ E.\ D.$$

Let X be a locally compact space. We assume conditions (iv), (vi), (vii) above and

(V)′ X is a left G-space.

Then we can prove that $K(V/G)$ and $K(X/G)$ are isomorphic ([3]).

Theorem 6. Under the above circumstance, $K_G(V)$ and $K_G(X)$ are isomorphic.

Proof. By the commutative

$$\begin{array}{ccc} K(V/G) & \xrightarrow{\approx} & K(X/G) \\ \tilde{\pi}^* \downarrow \approx & & \approx \downarrow \pi^* \\ K_G(V) & \longrightarrow & K_G(X) \end{array}$$

our assertion is abvious. *Q. E. D.*

References

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Hanyang University
Jeonbug National University