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On Weierstrass' and other examples of nowhere differentiable continuous function

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1. Introduction

A striking discovery of nowhere differentiable continuous functions was made by K. Weierstrass in 1872 [4]. His example was afforded by the lacunary Fourier series

$$W(x)=\sum_{n=0}^{\infty} a^n \cos (b^n \pi x) \tag{1}$$

where a and b were subject to the following conditions. :

ad a: $$0<a<1 \tag{2}$$

ad b: b is an odd interger with

$$ab>1+\frac{3}{2}\pi \tag{3}$$

This rather unnatural bound condition (3) was loosened by and by, and finally an extensive study on the behaviour of the partial sums of the series (1) was made by G. H. Hardy [1], who showed that under the fairly general situation:

ad b: b is any real number with

$$ab \geqq 1 \tag{4}$$

$W(x)$ is a nowhere differentiable continuous function. Some typical features of these functions are shown in verso.

After Weierstrass, other examples were constructed from different points of view. Here, the author would like to take up examples given by T. Takagi [2] and B. L. van der Waerden [3]. These examples are well resembled to the Weierstrass' example, and, so to say, more natural.

When one speaks of non-differentiability of a function, the shape as here exhibited below will immediately come up to his mind. The examples start from this function.

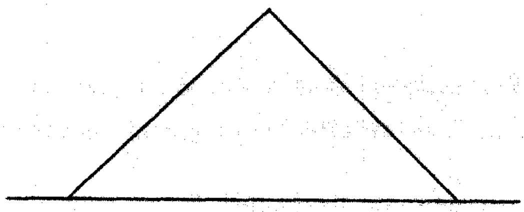


fig A.

We define

$$c(x)=\begin{cases} x & (0 \leqq x \leqq \frac{1}{2}) \\ 1-x & (\frac{1}{2} \leqq x \leqq 1) \end{cases} \tag{5}$$

and extend it periodically with period 1. Then, Takagi's example is

$$T(x)=\sum_{n=1}^{\infty}\frac{1}{2^n}\,c(2^n x)$$

and van der Waerden's example is

$$vdW(x)=\sum_{n=1}^{\infty}\frac{1}{10^n}\,c(10^n x)$$

Originally, these examples were not given under these forms, but it is not difficult to reformulate them in these forms. Now one sees that these functions are of the same type as (1), the difference exists only in that, in the latters, $c(x)$ is used instead of $\cos \pi x$. An amazing point of the Weierstrass's example lies in that it is constructed starting from a very well-behaved function. The graph of $T(x)$ is also given in verso. $vdW(x)$ is not very convenient to draw the graph.

The author would like to report in this paper an elementary way to proving that the function (1) is nowhere differentiable when b is subject to the following condition:

ad b* : *b is an integer with

$$ab \geqq 1 \tag{4}$$

Hardy's work is very exhaustive and it is almost a matter of experts. In his work, b could be any real number subject to the condition (4), it was not restricted to integers. In this general context, the series (1) is no more a Fourier series, and its analysis should really be left to the care of experts. In this work, b will be restricted to integers, but, as a compensation, our method can be applied to the analysis of the examples of the type of Takagi and van der Waerden. One would even observe that, applying our method, we can have examples of the same type starting from any continuously differentiable function having some minor properties.

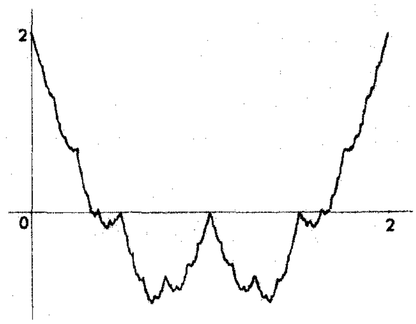


fig. 1 a=0. 5, b=2

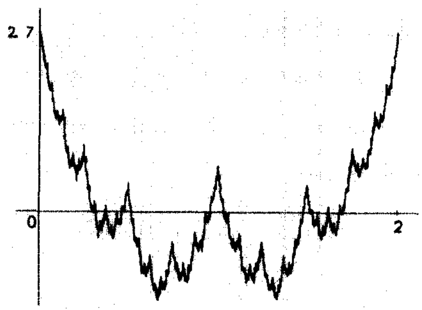


fig. 2 a=0. 63, b=2

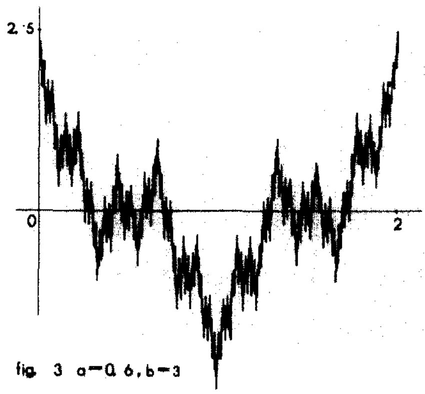


fig. 3 a=0. 6, b=3

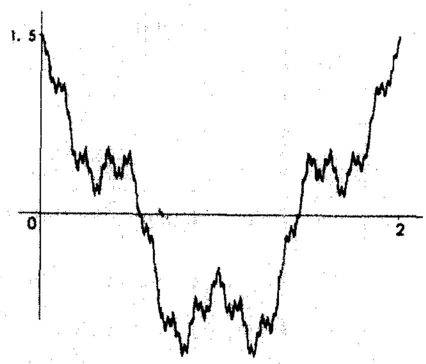
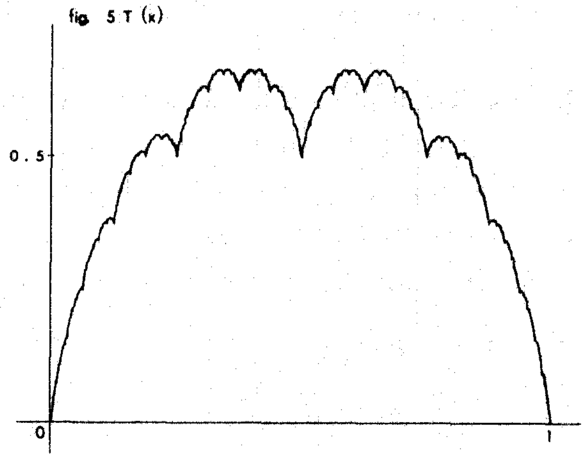


fig. 4 a=1 /3, b=4

fig. 5 T (x)



2. Main results

ASSUMPTIONS. We suppose f satisfies following conditions.

1. *f is continuously differentiable and periodic of period 2.*

2. *f is twice differentiable except at a finite number of points, and f'' is bounded with bound $\gamma < \frac{32}{3}$.*

3. *f takes its maximum value 1 at $x=0$, its minimum value -1 at $x=1$.* $f(\pm\frac{1}{2})=0$.

THEOREM. *Under these conditions, if*

ad a : $$0 < a < 1 \tag{1}$$

ad b : b is an integer with

$$ab \geq 1 \tag{4}$$

then

$$w(x)=\sum_{n=0}^{\infty} a^n f(b^n x) \tag{6}$$

is a nowhere differentiable continuous function.

PROOF. We show that $w(x)$ is not differentiable at x_0. To this end, we will estimate the amounts of variation of the difference quotient

$$\frac{w(x)-w(x_0)}{x-x_0}$$

near x_0, i.e. the variation of the slopes of the segments connecting the point P_0 of abscissa x_0 on the graph of $w(x)$ with points on the graph of $w(x)$ close to P_0.

Let m be a positive integer, and let k be an integer such that

$$k \leqq b^m x_0 < k+1.$$

k is supposed even, for otherwise we may replace the following consideration against $f(-x)$ resp. $-x_0$ in place of $f(x)$, x_0.

We take, along with P_0, three adjacent points P_1, P_2, P_3 with abscissae

$$x_1=\frac{k}{b^m}$$

$$x_2=\frac{k+(1/2)}{b^m}$$

$$x_3=\frac{k}{b^m}-\frac{1}{2b^{m+1}}$$

Now, to practice the evaluation, we divide the sum in (6) into two parts :

$$w(x)=u(x)+v(x)$$

$$u(x)=\sum_{n=0}^{m-1}a^n f(b^n x),\quad v(x)=\sum_{n=m}^{\infty}a^n f(b^n x)$$

Then

$$\begin{aligned}&|\text{ the slope of } P_1P_2-\text{the slope of } P_1P_3\ |\\ &=\left|\frac{w(x_2)-w(x_1)}{x_2-x_1}-\frac{w(x_3)-w(x_1)}{x_3-x_1}\right|\\ &=\left|\frac{u(x_2)-u(x_1)}{x_2-x_1}+\frac{v(x_2)-v(x_1)}{x_2-x_1}-\frac{u(x_3)-u(x_1)}{x_3-x_1}-\frac{v(x_3)-v(x_1)}{x_3-x_1}\right|\\ &\geqq\left|\frac{v(x_2)-v(x_1)}{x_2-x_1}-\frac{v(x_3)-v(x_1)}{x_3-x_1}\right|-\left|\frac{u(x_2)-u(x_1)}{x_2-x_1}-\frac{u(x_3)-u(x_1)}{x_3-x_1}\right|\end{aligned}$$

The following values are easily calculated. (Note that k is supposed even. We put $c=f(-\frac{1}{2b})$.)

	b : odd	b : even not a multiple of 4	b : even a multiple of 4
$v(x_1)$	$\frac{a^m}{1-a}$	$\frac{a^m}{1-a}$	$\frac{a^m}{1-a}$
$v(x_2)$	0	$-a^{m+1}+\frac{a^{m+2}}{1-a}$	$\frac{a^{m+1}}{1-a}$
$v(x_3)$	$a^m c$	$a^m c-a^{m+2}+\frac{a^{m+3}}{1-a}$	$a^m c+\frac{a^{m+2}}{1-a}$

Hence, putting

$$s_1=\frac{v(x_2)-v(x_1)}{x_2-x_1},\quad s_2=\frac{v(x_3)-v(x_1)}{x_3-x_1}$$

we have the following. For later use, we have put

$$s_2-s_1=d(ab)^m.$$

	b : odd	b : even not a multiple of 4	b : even a multiple of 4
s_1	$-\frac{2}{1-a}(ab)^m$	$-2(ab)^m+\frac{4}{b}(ab)^{m+1}$	$-2(ab)^m$
s_2	$2b(1-c)(ab)^m$ $+\frac{2}{1-a}(ab)^{m+1}$	$2b(1-c)(ab)^m$ $+2(ab)^{m+1}$ $+\frac{4}{b}(ab)^{m+2}$	$2b(1-c)(ab)^m+$ $2(ab)^{m+1}$
s_2-s_1	$2b(1-c)(ab)^m$ $+\frac{2}{1-a}(ab)^m$ $+\frac{2}{1-a}(ab)^{m+1}$	$2b(1-c)(ab)^m$ $+2(ab)^m$ $+(2+\frac{4}{b})(ab)^{m+1}$	$2b(1-c)(ab)^m$ $+2(ab)^m$ $+2(ab)^{m+1}$

		$+\frac{4}{b}(ab)^{m+2}$	
d	>4	$\geqq 8$ (when $b=2$) >4 (generally)	$\geqq 4$ (when $b=4$) >2 (generally)

On the other hand, using the mean value theorem, we have

$$\frac{u(x_2)-u(x_1)}{x_2-x_1}=u'(\xi_1),\quad \frac{u(x_3)-u(x_1)}{x_3-x_1}=u'(\xi_2)$$

for some points $\xi_1\in[x_1, x_2]$, $\xi_2\in[x_3, x_1]$. Then

$$\left|\frac{u(x_2)-u(x_1)}{x_2-x_1}-\frac{u(x_3)-u(x_1)}{x_3-x_1}\right|$$

$$=|u'(\xi_1)-u'(\xi_2)|$$

$$=\left|\int_{\xi_2}^{\xi_1} u''(x)\,dx\right|$$

$$\leqq\sum_{n=0}^{m-1}(ab^2)^n\left|\int_{\xi_2}^{\xi_1} f''(b^n x)dx\right|\leqq\gamma\sum_{n=0}^{m-1}(ab^2)^n(\xi_1-\xi_2)$$

$$=\gamma\frac{(ab^2)^m-1}{ab^2-1}\frac{1}{b^m}\left(\frac{1}{2}+\frac{1}{2b}\right)\leqq\gamma\frac{b+1}{2b(b-1)}(ab)^m$$

Combining these estimates, we have

$$|\text{the slope of } P_1P_2-\text{the slope of } P_1P_3|$$

$$\geqq\left(d-\gamma\frac{b+1}{2b(b-1)}\right)(ab)^m$$

$$=\frac{b+1}{2b(b-1)}\left(\frac{2b(b-1)}{b+1}d-\gamma\right)(ab)^m \tag{7}$$

Now the value of $\frac{2b(b-1)}{b+1}d$ is bounded from below in the following way.

b	2	3	4	b : odd $\geqq 5$	b : even $\geqq 6$
$\frac{2b+(b-1)}{b+1}d$ is larger than	$\frac{32}{3}$	12	$\frac{96}{5}$	$\frac{80}{3}$	$\frac{240}{7}$

Therefore, we see that, when

$$\gamma<\frac{32}{3}$$

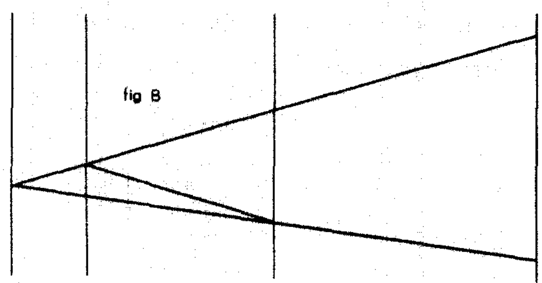
the difference of the slopes of P_1P_2 and P_1P_3 is larger than a fixed positive number e:

$$e>\frac{1}{2b}\left(\frac{32}{3}-\gamma\right)$$

independent of how the positive integer m is chosen.

Now we consider the slopes of P_oP_1. P_oP_2, P_oP_3, P_0 lies in the strip bounded by two virtical lines with abscissae $\frac{k}{b^m}$ and $\frac{k+1}{b^m}$, and inspecting each case when P_o is in the domains ①~⑥ shown in the figure, the difference of the slopes will be seen to be always larger than $\frac{1}{2(b+1)}e$.

	the difference of the slopes of the segments	is larger than
①	P_0P_1, P_0P_2	e
②	P_0P_2, P_0P_3	$\frac{1}{b+1}e$
③	P_0P_1, P_0P_3	$\frac{1}{b+1}e$
④	P_0P_1, P_0P_2	e
⑤	P_0P_1, P_0P_2	$\frac{1}{2(b+1)}e$
⑥	P_0P_1, P_0P_3	$\frac{1}{2(b+1)}e$



Thus we see that, in any small vicinity of x_0, the difference quotient oscillates in a range much broader than a fixed number $\frac{e}{2(b+1)}$, which implies that the difference quotient cannot have a definite limit, so the non-differentiability.

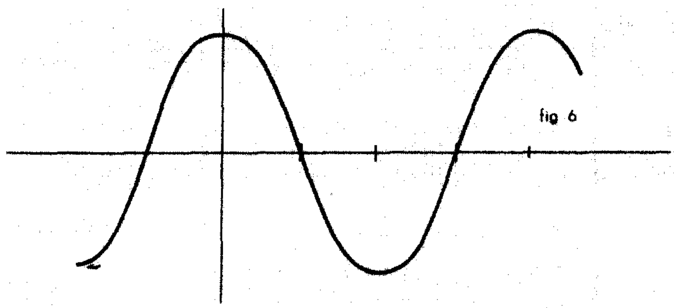
REMARK 1. For the Weierstrass' case, we take $f(x)=\cos \pi x$. Then $\gamma=\pi^2<\frac{32}{3}$, and our theorem applies.

Following function may also be a quite natural one of the same type.

$$f(x)=1-4x^2 \quad (|x|\leqq\frac{1}{2})$$

$$=4(x-1)^2-1 \quad (\frac{1}{2}\leqq x\leqq\frac{3}{2})$$

In this case, $\gamma=8$.



The condition $\gamma < \frac{32}{3}$ then appears to be rather restrictive, but, according to the estimate in (7), irrespective of how large γ is, the conclusion of our theorem is true for large b's.

REMARK 2. G. H. Hardy also showed that

$$W_1(x) = \sum_{n=1}^{\infty} a^n \sin (b^n \pi x)$$

is nowhere differentiable under the same conditions (1) and (4).

Our theorem cannot be applied to this case, but one sees without difficulty that the same reasoning can be used by choosing

$$x_1 = \frac{k}{b^m}$$

$$x_2 = \frac{k + (1/2)}{b^m}$$

$$x_3 = \frac{k+1}{b^m}$$

Putting

$$s_1 = \frac{v(x_2) - v(x_1)}{x_2 - x_1}, \qquad s_2 = \frac{v(x_3) - v(x_2)}{x_3 - x_2}$$

and $s_2 - s_1 = d\,(ab)^m$, we have the following values. For simplicity, we suppose again k even.

	b: even	$b \equiv 3 \pmod 4$	$b = 1 \pmod 4$
$v(x_1)$	0	0	0
$v(x_2)$	a^m	$\frac{a^m}{1+a}$	$\frac{a^m}{1-a}$
$v(x_3)$	0	0	0
$\lvert d \rvert$	4	$\frac{4}{1+a}$	$\frac{4}{1-a}$

Moreover, we have

$$\left| \frac{u(x_2)-u(x_1)}{x_2-x_1} - \frac{u(x_3)-u(x_2)}{x_3-x_2} \right| \leqq \gamma \frac{1}{b(b-1)} (ab)^m$$

Then, for $b \geqq 3$, we get the conclusion as before. For $b=2$, the choice of x_1, x_2, x_3 should be replaced as follows.

$$x_1 = \frac{k}{b^m} \qquad \text{then} \qquad v(x_1) = 0$$

$$x_2 = \frac{k+(1/3)}{b^m} \qquad\qquad v(x_2) = \sqrt{3}\,(1+\frac{a}{1+a})\,a^m \geq \frac{4\sqrt{3}}{3}a^m$$

$$x_3 = \frac{k+(1/2)}{b^m} \qquad\qquad v(x_3) = a^m$$

3. General case

We now state a similar theorem under the most general situation.

ASSUMPTIONS. We suppose a non constant function f satisfies following conditions.

1. *f is continuously differentiable and periodic of period* 1.
2. *f is twice differentiable except at a finite number of points, and f'' is bounded with bound γ.*

THEOREM. *Under these assumptions,*

$$w(x) = \sum_{n=0}^{\infty} a^n f(b^n x) \tag{6}$$

is nowhere differentiable if a is sufficiently small in the absolute value and b is sufficiently large in the absolute value. For example, we can conclude this when a, b are :

ad a: $$0 < |a| \leqq \frac{1}{4}$$

ad b: b is any real number with

$$|b| \geqq 3\gamma+1, \quad |ab| \geqq 1$$

PROOF. We proceed as before. We will show that $w(x)$ is not differentiable at x_0.

We may suppose the minimum value of f is 0, and the maximum value is 1. We assume

$$f(t_0)=0, \quad f(t_1)=1$$

and

$$0 \leqq t_0 \leqq t_1 \leqq 1$$

otherwise, consider $-f$. Note that according to the periodicity of f, $f(t_1-1)=1$

Let m be a positive integer, and let k be an integer such that

$$k \leqq b^m x_0 < k+1$$

k will be supposed even.

Now, choose

$$x_1 = \frac{1}{b^m}(k-1+t_1)$$

$$x_2 = \frac{1}{b^m}(k+t_0)$$

$$x_3 = \frac{1}{b^m}(k+t_1)$$

We divide the sum in (6) into $u(x)$ and $v(x)$ as before.

Then

$$v(x_1) = a^m f(b^m x_1) + a^{m+1} f(b^{m+1} x_1) + \cdots$$

$$= a^m + \sum_{n=m+1}^{\infty} a^n f(b^n x_1)$$

$$v(x_2) = \sum_{n=m+1}^{\infty} a^n f(b^n x_2)$$

$$v(x_3) = a^m + \sum_{n=m+1}^{\infty} a^n f(b^n x_3)$$

Therefore

$$\left| \frac{v(x_3)-v(x_2)}{x_3-x_2} - \frac{v(x_2)-v(x_1)}{x_2-x_1} \right|$$

$$= \left| a^m \left(\frac{1}{x_3-x_2} + \frac{1}{x_2-x_1}\right) + \sum_{n=m+1}^{\infty} a^n \left(\frac{f(b^n x_3)-f(b^n x_2)}{x_3-x_2} + \frac{f(b^n x_1)-f(b^n x_2)-1}{x_2-x_1}\right) \right|$$

$$\geqq \left(|a|^m - 2\sum_{n=m+1}^{\infty} |a|^n\right)\left(\frac{1}{x_3-x_2} + \frac{1}{x_2-x_1}\right)$$

$$= \left(|a|^m - \frac{2|a|^{m+1}}{1-|a|}\right)\left(\frac{1}{x_3-x_2} + \frac{1}{x_2-x_1}\right) \geqq \left(1 - \frac{2|a|}{1-|a|}\right) |ab|^m$$

The right hand side is $\geqq \frac{1}{3} |ab|^m$ when $0 \leqq |a| \leqq \frac{1}{4}$. On the other hand, from the estimate in the last section, we have

$$\left| \frac{u(x_3)-u(x_2)}{x_3-x_2} - \frac{u(x_2)-u(x_1)}{x_2-x_1} \right| \leqq \gamma \frac{|ab^2|^m - 1}{|ab^2|-1}(x_3-x_1)$$

$$\leqq \frac{\gamma}{|b|-1} |ab|^m$$

Therefore, when b is a real number satisfying

$$\frac{\gamma}{|b|-1} < \frac{1}{3}, \quad |ab| \geqq 1$$

we have, as in the proof of the last theorem,

$$|\text{ the slope of } P_1P_2 - \text{the slope of } P_1P_3 \,| \geqq e$$

$$\text{where} \quad e = \frac{1}{3} - \frac{\gamma}{|b|-1}$$

This will establish our theorem.

4. Takagi's and van der Waerden's case

Our theorem does not directly imply the result in the Takagi's and van der Waerden's case, where $f(x)$ is the function $c(x)$. But the same reasoning can be applied for

$$x_1 = \frac{k}{b^m}$$

$$x_2 = \frac{k+\frac{1}{2}}{b^m}$$

$$x_3 = \frac{k-\frac{1}{2}}{b^m}$$

k being supposed that $\frac{k}{b}$ is not a half-integer (i. e. not equal to an integer $+\frac{1}{2}$), otherwise we take $f(-x)$ and $-x_0$ as before.

The values of $v(x)$ is calcualted as before.

	b : odd	b : even
$v(x_1)$	0	0
$v(x_2)$ $v(x_3)$	$\frac{1}{2}\frac{a^m}{1-a}$	$\frac{1}{2}a^m$
s_1-s_2	$\frac{2}{1-a}(ab)^m$	$2(ab)^m$
	$\geqq 2$	$\geqq 2$

To calculate $\frac{u(x_2)-u(x_1)}{x_2-x_1}$, $\frac{u(x_3)-u(x_1)}{x_3-x_1}$, note that, if $\frac{k}{b^{m-n}}$ is neither an integer, nor a half-integer, the difference of $\frac{c(b^n x_2)-c(b^n x_1)}{x_2-x_1}$ and $\frac{c(b^n x_3)-c(b^n x_1)}{x_3-x_1}$ is zero because the interval $\left]\frac{k-\frac{1}{2}}{b^{m-n}}, \frac{k+\frac{1}{2}}{b^{m-n}}\right[$ contains neither an integer nor a half-integer, while, if it is an integer, the difference is $2b^n$.

If it happens by chance that $\frac{k}{b^{m-j}}$ is a half-integer, j is necessarily $<m-1$, b is even, and $\frac{k}{b^{m-n}}$ is an integer for $j<n\leqq m-1$, and the intervals $\left]\frac{k-\frac{1}{2}}{b^{m-n}}, \frac{k+\frac{1}{2}}{b^{m-n}}\right[$ with $n<j$ contain no more half-integers. From these, we conclude that

$$\frac{u(x_2)-u(x_1)}{x_2-x_1}-\frac{u(x_3)-u(x_1)}{x_3-x_1}\geqq 0.$$

Therefore

$$\frac{w(x_2)-w(x_1)}{x_2-x_1}-\frac{w(x_3)-w(x_1)}{x_3-x_1}$$

$$=\frac{u(x_2)-u(x_1)}{x_2-x_1}-\frac{u(x_3)-u(x_1)}{x_3-x_1}+\frac{v(x_2)-v(x_1)}{x_2-x_1}-\frac{v(x_3)-v(x_1)}{x_3-x_1}\geqq 2.$$

The rest of the arguments are the same as before.

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