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SCALE-INVARIANT MEASURABILITY IN YEH-WIENER SPACE

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1. Introduction

Let $R=\{(s,t) : a\leq s\leq b, \alpha\leq t\leq\beta\}$ and $C_2[R]$ be Yeh-Wiener space, i.e. $C_2[R]=\{x(\cdot, \cdot) : x(a,t)=x(s, \alpha)=0, x(s, t)$ is continuous on $R\}$. $C_2[R]$ is often referred to as two parameter Wiener space. Let $a=s_0<s_1<\cdots<s_m=b$ and $\alpha=t_0<t_1<\cdots<t_n=\beta$ and let $-\infty\leq a_{j,k}\leq b_{j,k}\leq+\infty$ be given for $j=1, 2, \ldots, m$ and $k=1, 2, \ldots, n$. Let $E=(a_{11}, b_{11}]\times\ldots\times(a_{mn}, b_{mn}]$. $I=J_{(\vec{s},\vec{t})}(E)\equiv\{x\in C_2[R] ; (x(s_1, t_1), \ldots, x(s_m, t_n))\in E\}$ is called a strict interval of $C_2[R]$. If E is an arbitrary measurable subset of $\mathbf{R}^{mn}$, then I is called an interval of $C_2[R]$.

The collection $\mathcal{J}$ of all such strict intervals form a semi-algebra of subsets of $C_2[R]$. The measure of the strict interval I is defined to be

$$m_1(I)=\int_E\omega(\vec{u} : \vec{s} : \vec{t})d\vec{u},$$

where

$$\begin{aligned}\omega(\vec{u} : \vec{s} : \vec{t}) &=\omega(u_{11}, \ldots, u_{mn} : s_1, \ldots, s_m : t_1, \ldots, t_n)\\ &=\prod_{j=1}^{m}\prod_{k=1}^{n}\{\pi(s_j-s_{j-1})(t_k-t_{k-1})\}^{-1/2}\\ &\quad\cdot\exp\left\{\frac{-(u_{jk}-u_{j-1,k}-u_{j,k-1}+u_{j-1,k-1})^2}{(s_j-s_{j-1})(t_k-t_{k-1})}\right\}\end{aligned}$$

and $u_{0,k}=u_{j,0}=u_{0,0}=0$ for all j and k. This measure is countably additive on $\mathcal{J}$ and can be extended in the usual way to the σ-algebra $\sigma(\mathcal{J})$ generated by the strict intervals and then can be further extended so as to be a complete measure. This completed measure space is denoted by $(C_2[R], \mathcal{Y}_1, m_1)$ and $\mathcal{Y}_1$ is called the class of Yeh-Wiener measurable sets.

For $x\in C_2[R]$, let $\|x\|=\max_{(s,t)\in R}|x(s,t)|$. Then $(C_2[R], \|\cdot\|)$ is a separable Banach space.

Let $\mathcal{B}$ be the collection of all sets of the form $J_{(\vec{s},\vec{t})}(B)$ for all $(\vec{s} ; \vec{t})$ and all Borel set B in $\mathbf{R}^{Ln}$. Then $\mathcal{B}$ is an algebra of subsets of $C_2[R]$. Let $\sigma(\mathcal{B})$ be the σ-agebra generated by $\mathcal{B}$ and $\mathcal{B}(C_2[R])$ be the class of Borel sets in $C_2[R]$. Then it is well known that $\sigma(\mathcal{J})=\sigma(\mathcal{B})=\mathcal{B}(C_2[R])$. $\sigma(\mathcal{J})$ is

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sometimes referred to as the σ-algebra of strictly Yeh-Wiener measurable sets.

Let σ_m be the partition:

$$\sigma_m=\{(s_j,\ t_k)\ :\ s_j=a+\frac{j(b-a)}{m},\ \ t_k=\alpha+\frac{k(\beta-\alpha)}{m}\ :j,\ k=1,\ 2,\ \ldots,\ m\}.$$

For each $x\in C_2[R]$, let

$$S_{\sigma_m}(x)=\sum_{j=1}^{m}\sum_{k=1}^{m}\{x(s_j,\ t_k)-x(s_{j-1},\ t_k)-x(s_j,\ t_{k-1})+x(s_{j-1},\ t_{k-1})\}^2.$$

For each $\lambda\geq 0$, let

$$C_\lambda=\{x\in C_2[R]\ :\ \lim_{n\to\infty}S_{\sigma_{2^n}}(x)=\lambda^2(b-a)(\beta-\alpha)/2\}$$

$$D=\{x\in C_2[R]\ :\ \lim_{n\to\infty}S_{\sigma_{2^n}}(x)\ \text{fails to exist}\}.$$

Note that $\nu C_\lambda=C_{\nu\lambda}$ for $\nu>0$, $\lambda\geq 0$. Clearly $C_\lambda(\lambda\geq 0)$ and D are Borel sets and $C_2[R]$ is the disjoint union of this family of sets.

The key to our discussion is the following result due to Skoug [4].

THEOREM 1.1. $m_1(C_1)=1$.

In §2 we will extend this result to partitions $\sigma_{h(n)}$ where h is an increasing function from $\mathbf{N}$ into $\mathbf{N}$ such that $n\leq h(n)$ for all $n\in\mathbf{N}$.

DEFITIONS. A set $E\subseteq C_2[R]$ is said to be scale-invariant measurable if $\lambda E\in\mathscr{Y}_1$ for every $\lambda>0$. A scale-invariant measurable set N is called scale-invariant null if $m_1(\lambda N)=0$ for every $\lambda>0$. A property which holds except on a scale-invariant null set will be said to hold s-almost everywhere (denoted by s-$a.e.$).

In this paper we will extend the results on scale-invariant measurability in Wiener space which Johnson and Skoug obtained in [2] to Yeh-Wiener space. Many of the concepts, theorems and proofs will be much like analogous results in [2]. A number of the proofs will be omitted.

2. Preliminaries and Some Results in Yeh-Wiener Space

The following three propositions are well known results. We will state them without proof.

PROPOSITION 2.1. *E is Lebesgue measurable in* R^{mn} *iff* $J_{(\vec{s},\vec{t})}(E)$ *is Yeh-Wiener measurable. In this case,*

$$m_1(J_{(\vec{s},\vec{t})}(E))=\int_E\omega(\vec{u}\ :\ \vec{s}\ :\ \vec{t})d\vec{u}.$$

PROPOSITION 2.2. *Let $f(u_{11}, \ldots, u_{mn})$ be a Lebesgue measurable function on $\mathbf{R}^{mn}$ and $F(x)=f(x(s_1, t_1), \ldots, \times(s_m, t_n))$. Then F is Yeh-Wiener measurable and*

$$\int_{C_2[R]} F(x)dm_1(x) \overset{*}{=} \int_{\mathbf{R}^{mn}} f(\vec{u})\omega(\vec{u} : \vec{s} : \vec{t})d\vec{u}.$$

Note that actually $F(x)$ is Yeh-Wiener measurable iff f is Lebesgue measurable.

PROPOSITON 2.3. (a) *If E is Yeh-Wiener measurable, then $-E$ is Yeh-Wiener measurable and $m_1E=m_1(-E)$.*

(b) $$\int_{C_2[R]} F(x)dm_1(x)=\int_{C_2[R]} F(-x)dm_1(x).$$

Since $\sigma(\mathcal{B})=\mathcal{B}(C_2[R])$ we have that if E is a Borel set in $\mathbf{R}^{mn}$, then $J_{(\vec{s}, \vec{t})}(E)$ is a Borel set in $C_2[R]$. The following proposition shows the converse to this fact. First of all we state a simple lemma.

LEMMA 2.4. *Given any real numbers u_{ij}, $0\leq i\leq m$, $0\leq j\leq n$, let udenote the matrix (u_{ij}). Then there exists a piecewise linear continuous function $H(u)$ on R such that $H(u)(s_i, t_j)=u_{ij}$; further, if $u_{ij}^{(k)} \to u_{ij}$ as $k \to \infty$ for $0\leq i\leq m$, $0\leq j\leq n$, $H(u^{(k)}) \to H(u)$ uniformly on R.*

PROPOSITION 2.5. *If $J_{(\vec{s},\vec{t})}(E)$ is a Borel set in $C_2[R]$, then E is a Borel set in R^{mn}.*

Proof. Define H on $\mathbf{R}^{mn}$ as in Lemma 2.4 so that $H(u)(s, t)=0$ if $s=a$ or $t=\alpha$. Such an H is a continuous (and hence Borel) function from $\mathbf{R}^{mn}$ to $C_2[R]$. Now $X_E(u)=(X_{J(\vec{s}, \vec{t})(E)}\circ H)(u)$ since $u\in E$ iff $H(u)\in J_{(\vec{s}, \vec{t})}(E)$. Suppose $J_{(\vec{s}, \vec{t})}(E)$ is a Borel set in $C_2[R]$. Then $X_E=X_{J(\vec{s}, \vec{t})(E)}\circ H$ is a Borel function since it is the composition of two Borel functions. Hence E is a Borel subset of $\mathbf{R}^{mn}$.

PROPOSITION 2.6. *Let $h : N \to N$ be an increasing function such that $n\leq h(n)$ for all $n\in N$. Let*

$$C_\lambda{}^h= \{x\in C_2[R] : \lim_{n\to\infty} S_{\sigma_{h(n)}}(x)=\lambda^2(b-a)(\beta-\alpha)/2\}.$$

Then $m_1(C_1{}^h)=1$.

Proof. Skoug [4, Proof of Lemma 1] showed that

$$\int_{C_2[R]} \{S_{\sigma h(n)}(x)-(b-a)(\beta-\alpha)/2\}^2dx$$
$$=1/2\{(b-a)(\beta-\alpha)/h(n)\}^2.$$

Let $E_n= \{x : |S_{\sigma h(n)}(x)-(b-a)(\beta-\alpha)/2|\geq \frac{\log n}{\sqrt{2n}}(b-a)(\beta-\alpha)\}$.

$$1/2\left\{\frac{(b-a)(\beta-\alpha)}{h(n)\}}\right\}^2=\int_{C_2[R]}\left\{S_{\sigma_{h(n)}}(x)-\frac{(b-a)(\beta-\alpha)}{2}\right\}^2dx$$
$$\geq\int_{E_n}\left\{S_{\sigma_{h(n)}}(x)-\frac{(b-a)(\beta-\alpha)}{2}\right\}^2dx$$
$$\geq\frac{(\log n)^2}{2n}(b-a)^2(\beta-\alpha)^2\cdot m_1(E_n).$$

Hence $m_1(E_n)\leq\frac{n}{[h(n)\log n]^2}\leq\frac{1}{n(\log n)^2}$.

Let $F_n=\bigcup_{k=n}^{\infty}E_k$ and $F=\bigcap_{n=1}^{\infty}F_n$. Then

$$m_1(F)\leq m_1(F_n)\leq\sum_{k=n}^{\infty}m_1(E_k)\leq\sum_{k=n}^{\infty}\frac{1}{k(\log k)^2}\to 0\quad\text{as } n\to\infty.$$

So $m_1(F)=0$. But for $x\notin F$, i. e. for $x\notin E_k$ for all $k\geq n$ and for some n,

$$\left|S_{\sigma_{h(k)}}(x)-\frac{(b-a)(\beta-\alpha)}{2}\right|<\frac{\log k}{\sqrt{2k}}(b-a)(\beta-\alpha)\ \text{for all } k\geq n.$$

Hence $\lim_{k\to\infty}\left|S_{\sigma_{h(k)}}(x)-\frac{(b-a)(\beta-\alpha)}{2}\right|\leq\lim_{k\to\infty}\frac{\log k}{\sqrt{2k}}(b-a)(\beta-\alpha)=0.$

This implies that $\lim_{k\to\infty}S_{\sigma_{h(k)}}(x)=\frac{(b-a)(\beta-\alpha)}{2}$ for $x\notin F$. But $m_1(F)=0$.

3. Scale-Invariant Measurable Sets in Yeh-Wiener Space

Let m_λ be the Borel measure given by $m_\lambda(B)=m_1(\lambda^{-1}B)$ for $B\in\mathcal{B}(C_2[R])$. Since $\lambda^{-1}C_\lambda=C_1$, $m_\lambda(C_\lambda)=m_1(C_1)=1$ by Theorem 1.1.

Let $\mathcal{Y}_\lambda$ denote the σ-algebra obtained by completing $(C_2[R], \mathcal{B}(C_2[R], m_\lambda)$ and let $\mathcal{N}_\lambda$ be the class of m_λ-null sets. Note that every subset of $C_2[R]\backslash C_\lambda$ is in $\mathcal{N}_\lambda$. Let $\mathcal{Y}$ and $\mathcal{N}$ be the class of scale-invariant measurable sets and scale-invariant null sets, respectively.

PROPOSITION 3.1. (i) *N is in $\mathcal{N}_\lambda$ iff $\lambda^{-1}N$ is in $\mathcal{N}_1$; equivalently, $\mathcal{N}_\lambda=\lambda\mathcal{N}_1$.*
(ii) *E is in $\mathcal{Y}_\lambda$ iff $\lambda^{-1}E$ is in $\mathcal{Y}_1$; equivalently, $\mathcal{Y}_\lambda=\lambda\mathcal{Y}_1$.*
(iii) *$m_\lambda(E)=m_1(\lambda^{-1}E)$ for E in $\mathcal{Y}_\lambda$.*

Proof. (i) Let N be in $\mathcal{N}_\lambda$. Then $N\subset M$ where M is an m_λ-null Borel set. Hence $m_1(\lambda^{-1}M)=m_\lambda M=0$ and so $\lambda^{-1}M$ is an m_1-null Borel set. But then $\lambda^{-1}N\subset\lambda^{-1}M$ is in $\mathcal{N}_1$. The converse can be shown in essentially the same way.

(ii) Let E be in $\mathcal{Y}_\lambda$. Then $E=B\cup N$ where B is in $\mathcal{B}(C_2[R])$ and N is in $\mathcal{N}_\lambda$. Then $\lambda^{-1}N$ is in $\mathcal{N}_1$ by (i) and so $\lambda^{-1}E=\lambda^{-1}B\cup\lambda^{-1}N$ is in $\mathcal{Y}_1$. The rest of (ii) is easily checked.

(iii) Let E be in $\mathcal{Y}_\lambda$. Then $E=B\cup M$ where B is in $\mathcal{B}(C_2[R])$ and N is m_λ-null. Then

$$m_\lambda(E)=m_\lambda(B\cup N)=m_\lambda(B)=m_1(\lambda^{-1}B)=m_1(\lambda^{-1}B\cup\lambda^{-1}N)=m_1(\lambda^{-1}E).$$

PROPOSITION 3. 2. $\mathscr{Y}=\bigcap_{\lambda>0}\mathscr{Y}_\lambda$; $\mathscr{N}=\bigcup_{\lambda>0}\mathscr{N}_\lambda$; *$\mathscr{Y}$ is a σ-algebra of subsets of $C_2[R]$.*

REMARK. Beginning with this proposition, most of the proofs in the rest of this section are much like the proofs of corresponding results in [2]. We will include a few of these proofs but will omit most of them.

PROPOSITION 3. 3. (i) *E is in $\mathscr{Y}$ iff $E\cap C_\lambda$ is in $\mathscr{Y}_\lambda$ for every $\lambda>0$.*
(ii) *E is in $\mathscr{N}$ iff $E\cap C_\lambda$ is in $\mathscr{N}_\lambda$ for every $\lambda>0$.*

The next theorem is quite simple. But it gives a very useful characterization of $\mathscr{Y}$ and $\mathscr{N}$ in that it shows rather well what scale-invariant measurable sets and scale-invariant null sets are really like and how they compare to Yeh-Wiener measurable sets and Yeh-Wiener null sets respectively.

THEOREM 3. 4. (i) *E is in $\mathscr{Y}$ iff E has the form*

$$E=(\bigcup_{\lambda<0}E_\lambda)\cup L, \tag{3.1}$$

where each E_λ is an m_λ-measurable subset of C_λ and L is an arbitrary subset of $C_0\cup D$. Further, for E written in this manner, $m_\lambda(E)=m_\lambda(E_\lambda)$ for all $\lambda>0$.
(ii) *N is in $\mathscr{N}$ iff N has the form*

$$N=(\bigcup_{\lambda>0}N_\lambda)\cup L, \tag{3.2}$$

where each N_λ is an m_λ-null subset of C_λ and L is an arbitrary subset of $C_0\cup D$.

REMARK. The preceeding theorem shows that there are many more Yeh-Wiener measurable sets than scale-invariant measurable sets: A set E is Yeh-Wiener measurable if and only if it has the form $E_1\cup L$ where E_1 is an m_1-measurable subset of C_1 and L is an arbitrary subset of $(\bigcup_{0<\lambda\neq1}C_\lambda)\cup D\cup C_0$. Similarly a set is Yeh-Wiener null if and only if it has the form $N_1\cup L$ where N_1 is an m_1-null subset of C_1 and L is an arbitrary subset of $(\bigcup_{0<\lambda\neq1}C_\lambda)\cup D\cup C_0$.

Let $a=s_0<s_1<\ldots<s_m=b$, $\alpha=t_0<t_1<\ldots<t_n=S$ and let E be any subset of $\mathbf{R}^{mn}$. Let

$$Q=J_{(\vec{s},\vec{t})}(E)=\{x\in C_2[R]:(x(s_1,t_1),\ \ldots,\ x(s_m,t_n))\in E\}. \tag{3.3}$$

We have seen, in §2, that E is Borel measurable in $\mathbf{R}^{mn}$ if and only if Q is Borel measurable in $C_2[R]$ and that E is Lebesgue measurable in $\mathbf{R}^{mn}$ if and only if Q is Yeh-Wiener measurable [3]. It is easy to see that such sets Q are scale-invariant measurable, since for any $\lambda>0$,

$$\lambda Q=\{x\in C_2[R]:(x(s_1,t_1),\ \ldots,\ x(s_m,t_n))\in\lambda^{-1}E\}$$

is Yeh-Wiener measurable.

PROPOSITION 3.5. *For every* $\lambda_0>0$, $\mathcal{B}(C_2[R])\underset{\neq}{\subset}\mathcal{Y}\underset{\neq}{\subset}\mathcal{Y}_{\lambda_0}$.

The following result of Skoug [4] becomes rather transparent using Theorem 3.4.

COROLLARY 3.6. *Let f be any function with domain* $(0, \infty)$ *and satisfying* $0\leq f(\lambda)\leq 1$. *Then there exists E in $\mathcal{Y}$ such that $m_1(\lambda E)=f(\lambda)$ for all $\lambda>0$.*

Proof. For each $\lambda>0$, pick $E_\lambda\subset C_\lambda$ such that E_λ is in $\mathcal{Y}_\lambda$ and $m_\lambda(E_\lambda)=f(\lambda^{-1})$. (Such E_λ exists by the following lemma.) Then $E\equiv\bigcup_{\lambda>0}E_\lambda$ is the desired set since, by Proposition 3.1 and Theorem 3.4, we have $m_1(\lambda E)=m_{\lambda^{-1}}(E)=m_{\lambda^{-1}}(E_{\lambda^{-1}})=f(\lambda)$.

LEMMA. *Given $\gamma\in[0,1]$, there exists $E_\lambda\subset C_\lambda$ such that $E_\lambda\in\mathcal{Y}_\lambda$ and $m_\lambda(E_\lambda)=\gamma$ for each $\lambda>0$.*

Proof. Given $\gamma\in[0,1]$, there exists a real number a_γ such that.

$$\frac{1}{\sqrt{\pi(b-a)(\beta-\alpha)}}\int_{-\infty}^{a_\gamma}e^{-\frac{u^2}{(b-a)(\beta-\alpha)}}du=\gamma.$$

Let $E=\{x\in C_2[R]:-\infty<x(b,\beta)\leq a_\gamma\}$. Then E is in $\mathcal{Y}_1$ and $m_1(E)=\gamma$. Let $E_1=E\cap C_1$. Then $E_1\in\mathcal{Y}_1$ and $m_1(E_1)=m_1(E)=\gamma$. Let $E_\lambda=\lambda E_1$. Then E_λ is in $\mathcal{Y}_\lambda$ and $E_\lambda\subset\lambda C_1=C_\lambda$ and $m_\lambda(E_b)=m_\lambda(\lambda E_1)=m_1(E_1)=\gamma$.

Our sets C_λ, $\lambda\geq 0$ and D depend on the particular sequence of partitions on R that we choose. If $\sigma_{h(n)}$ denotes another sequence of partitions, we may let

$$C_\lambda^h\equiv\{x\in C_2[R]:\lim_{n\to\infty}S_{\sigma_{h(n)}}(x)=\lambda^2(b-a)(\beta-\alpha)/2\}$$

and

$$D^h\equiv\{x\in C_2[R]:\lim_{n\to\infty}S_{\sigma_{h(n)}}(x)\text{ fails to exist}\}.$$

Essentially because of Proposition 2.6, all of the results obtained up to this point, with changes in notation where appropriate, go through. Note, however, that $\mathcal{Y}_\lambda$, $\mathcal{U}_\lambda$, m_λ, $\mathcal{Y}$ and $\mathcal{U}$ are all independent of the sequence of partitions. A set E in $\mathcal{Y}$ now has two decompositions according to the two versions of Theorem 3.4:

$$E=(\bigcup_{\lambda>0}E_\lambda)\cup L=(\bigcup_{\lambda>0}E_\lambda^h)\cup L^h \tag{3.4}$$

where $E_\lambda^h=E\cap C_\lambda^h$ and $L^h=E\cap(C_0^h\cup D^h)$. How do these two decompositions relate to one another? The next proposition shows that they agree up to a scale-invariant null set.

PROPOSITION 3.7. *The two decompositions of E given by* (3.4) *have the property that the set*

$$(3.5) \qquad (\bigcup_{\lambda>0} E_\lambda \Delta E_\lambda^h) \cup (L \Delta L^h)$$

is scale-invariant null.

Proof. First note that for all $\lambda>0$

$$\begin{aligned} m_\lambda(E_\lambda \backslash E_\lambda^h) &= m_\lambda[(E \cap C_\lambda) \backslash (E \cap C_\lambda^h)] \\ &= m_\lambda[E_\lambda \cap (C_\lambda \backslash C_\lambda^h)] \\ &\leq m_\lambda(C_\lambda \backslash C_\lambda^h) \\ &\leq m_\lambda(C_2[R] \backslash C_\lambda^h) = 0. \end{aligned}$$

Thus by Theorem 3.4, the set $\bigcup_{\lambda>0}(E_\lambda \backslash E_\lambda^h) \cup (L \backslash L^h)$ is scale-invariant null. In similar fashion one cans how that the set $\bigcup_{\lambda>0}(E_\lambda^h \backslash E_\lambda) \cup (L^h \backslash L)$ is scale-invariant null which concludes the proof since

$$\bigcup_{\lambda>0}(E_\lambda \Delta E_\lambda^h) \cup (L \Delta L^h) = \{\bigcup_{\lambda>0}(E_\lambda \backslash E_\lambda^h) \cup (L \backslash L^h)\} \cup \{\bigcup_{\lambda>0}(E_\lambda^h \backslash E_\lambda) \cup (L^h \backslash L)\}.$$

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