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GENERIC SUBMANIFOLDS WITH PARALLEL RICCI CURVATURE OF $S^{2m+1}(1)$

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0. Introduction

A submanifold M of a Sasakian manifold M^{2m+1} is called a generic (an *antiholomorphic*) if the normal space $N_p(M)$ of M at any point $p\in M$ is mapped into the tangent space $T_p(M)$ by action of the structure tensor F of the ambient manifold M^{2m+1}, that is, $FN_p(M)\subset T_p(M)$ for each point $p\in M$ ([3], [9], [10]).

The main purpose of the present paper is to characterize generic submanifolds with parallel Ricci tensor of an odd-dimensional sphere $S^{2m+1}(1)$ such that the Sasakian structure vector is tangent to the submanifold.

In characterizing the submanifolds, we shall use the following Theorem A and B.

TEOREM A([7]). *Let M be an n-dimensional complete generic submanifold with flat normal connection of an odd-dimensional unit sphere $S^{2m+1}(1)$ and let the Sasakian structure vector defined on $S^{2m+1}(1)$ be tangent to M. If the structure induced on M is normal and if the mean curvature vector of M is parallel in the normal bundle, then M is a pythagorean product of the form*

$$S^{p_1}(r_1)\times\cdots\cdots\times S^{p_N}(r_N),$$

where $p_1, \ldots, p_N$ are odd-numbers $\geqq 1$, $r_1{}^2+\cdots\cdots+r_N{}^2=1$, $N=2m+1-n$, $S^p(r)$ being p-dimensional sphere with radius $r>0$.

THEOREM B([4]). *Let M be an n-dimensional complete generic submanifolds of an odd-dimensional sphere $S^{2m+1}(1)$ with flat normal connection. Suppose that M is proper Einstein and the scalar curvature K of M satisfies $K\neq n(n-1)$. If the induced structure on M is partially integrable, then M is of the form*

$$S^m(r)\times S^m(r),\quad S^q(r)\times\cdots\cdots\times S^q(r)\ (N\text{-times}),\quad N_q=n,$$

where q is an odd-number and $2m-n+2=N$, $(N\neq n+2)$.

1. Preliminaries

Let M^{2m+1} be a $(2m+1)$-dimensional Sasakian manifold covered by a system of coordinate neighborhoods $\{U\,;\,y^h\}$ and with structure tensors $(F_j{}^h,\ G_{ji},\ V^h)$.

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Then we have

$$(1.1)\qquad \begin{cases} F_j{}^hF_i{}^j=-\delta_i{}^h+V_iV^h, & V_jF_i{}^j=0, \quad F_i{}^hV^i=0, \\ V_iV^i=1, & G_{hk}F_j{}^hF_i{}^k=G_{ji}-V_jV_i, \end{cases}$$

V_i being the associated 1-form of V^h, where here and in the sequal the indices $h, i, j, \ldots$ run over the range $\{1, 2, \ldots, (2m+1)\}$.

We denote by ∇_i the operator of covariant differentiation with respect to the Christoffel symbols $\{{}_i{}^h{}_j\}$ formed with G_{ji}. We then have

$$(1.2)\quad \nabla_iV^h=F_i{}^h, \quad \nabla_jF_i{}^h=-G_{ji}V^h+\delta_j{}^hV_i.$$

Let M be an n-dimensional Riemannian manifold covered by a system of coordinate neighborhoods $\{V\,;\,x^a\}$ and isometrically immersed in M^{2m+1} by the immersion $i: M\rightarrow M^{2m+1}$. We identify $i(M)$ with M itself and represent the immersion locally by $y^h=y^h(x^a)$, where here and in the throughout this paper the indices a, b, c, d and e run over the range $\{1, 2, 3, \ldots, n\}$. If we put $B_a{}^h=\partial_ay^h$, $(\partial_a=\partial/\partial x^a)$, then $B_a{}^h$ are n linearly independent vectors of M^{2m+1} tangent to M.

We denote by $C_x{}^hp$ $(=2m+1-n)$ mutually orthogonal unit normals to M, then we have

$$(1.3)\qquad g_{cb}=G_{ji}B_c{}^jB_b{}^i, \quad G_{ji}B_a{}^jC_x{}^i=0, \quad g_{yx}=G_{ji}C_y{}^jC_x{}^i.$$

Since the immersion is isometric, g_{cb} and g_{yx} being the fundamental metric tensor of M and that of the normal bundle of M respectively. Throughout this paper the indices x, y, z, u, v and w run over the range $\{1^*, 2^*, \ldots, p^*\}$.

Denoting by ∇_c the operator of van der Waerden–Borotolotti covariant differentiation with respect to the Christoffel symbols $\{{}_b{}^a{}_c\}$ formed with g_{cb}, then we obtain equations of the Gauss and Weingarten for M

$$(1.4)\qquad \nabla_cB_b{}^h=h_{cb}{}^xC_x{}^h, \quad \nabla_cC_x{}^h=-h_c{}^a{}_xB_a{}^h$$

respectively. $h_{cb}{}^x$ and $h_c{}^a{}_x$ appearing here are both called the second fundamental form of M with respect to the normals $C_x{}^h$ and are related by $h_c{}^a{}_x=h_{cb}{}^yg^{ba}g_{yx}$, $(g^{cb})=(g_{cb})^{-1}$.

The mean curvature vector $\frac{1}{n}h^x=\frac{1}{n}g^{cb}h_{cb}{}^x$ of M is said to be *parallel* if $\nabla_ch^x=0$.

If the ambient manifold M^{2m+1} is a $(2m+1)$-dimensional unit sphere $S^{2m+1}(1)$, then the equations of Gauss, Codazzi and Ricci for M are given respectively by

$$(1.5)\qquad K_{dcb}{}^a=\delta_d{}^ag_{cb}-\delta_c{}^ag_{db}+h_d{}^{ax}h_{cbx}-h_c{}^{ax}h_{dbx},$$

$$(1.6)\qquad \nabla_dh_{cb}{}^x-\nabla_ch_{db}{}^x=0,$$

$$(1.7)\qquad K_{dcy}{}^x=h_{de}{}^xh_c{}^e{}_y-h_{ce}{}^xh_d{}^e{}_y,$$

where $K_{dcb}{}^a$ and $K_{dcy}{}^x$ are the curvature tensor of M and that of the connection in the normal bundle respectively.

If $K_{dcy}{}^x=0$, that is,

$$h_{de}{}^x h_c{}^e{}_y = h_{ce}{}^x h_d{}^e{}_y, \tag{1.8}$$

then the normal connection of M is said to be *flat*.

We have from (1.5)

$$K_{cb} = (n-1)g_{cb} + h_x h_{cb}{}^x - h_{ce}{}^x h_b{}^e{}_x, \tag{1.9}$$

K_{cb} being the Ricci tensor of M, which implies

$$K = n(n-1) + h_x h^x - h_{cb}{}^x h^{cb}{}_x \tag{1.10}$$

where K is the scalar curvature of M. If $\nabla_d K_{cb}=0$, then the submanifold M is said to be *Ricci parallel*.

A submanifold M is called *proper Einstein* if it satisfies $K_{cb}=(K/n)g_{cb}$, $K\neq 0$.

Throughout this paper, we consider only generic submanifolds immersed in Sasakian manifold. Then we can put in each coordinate neighborhood

$$F_j{}^h B_c{}^j = f_c{}^a B_a{}^h - f_c{}^x C_x{}^h, \tag{1.11}$$

$$F_j{}^h C_x{}^j = f_x{}^a B_a{}^h, \tag{1.12}$$

$$V^h = f^a B_a{}^h + f^x C_x{}^h, \tag{1.13}$$

where $f_c{}^a$ is a tensor field of type (1, 1), $f_c{}^x$ a local 1-form for each fixed index x, f^a a vector field and f^x a function for each fixed index x.

Now applying the operator F to (1.11)–(1.13) and taking account of (1.1), (1.3) and these equations, we essily that ([3])

$$\begin{cases} f_c{}^e f_e{}^a = -\delta_c{}^a + f_c{}^x f_x{}^a + f_c f^a, \quad f_c{}^e f_e{}^x = -f_c f^x, \\ f^e f_e{}^a = -f^x f_x{}^a, \quad f_x{}^e f_e{}^y = \delta_x{}^y - f_x f^y, \quad f^e f_e{}^x = 0, \\ f_e f^e + f_x f^x = 1, \quad g_{de} f_c{}^d f_b{}^e = g_{cb} - f_c{}^x f_{bx} - f_c f_b, \end{cases} \tag{1.14}$$

where $f_c = f^e g_{ce}$ and $f_x = f^y g_{yx}$.

Putting $f_{cb} = f_c{}^a g_{ab}$ and $f_{xc} = f_x{}^a g_{ac}$, then we have

$$f_{cb} = -f_{bc}, \quad f_{xc} = f_{cx}.$$

Differentiating (1.11)–(1.13) covariantly along M and making use of (1.2), (1.4) and these relationships, we find ([3])

$$\nabla_c f_b{}^a = -g_{cb} f^a + \delta_c{}^a f_b + h_{cb}{}^x f_x{}^a - h_c{}^a{}_x f_b{}^x, \tag{1.15}$$

$$\nabla_c f_b{}^x = g_{cb} f^x + h_{ce}{}^x f_b{}^e. \tag{1.16}$$

$$\nabla_c f_b = f_{cb} + h_{cb}{}^x f_x, \tag{1.17}$$

$$\nabla_c f_x = -f_{cx} - h_c{}^e{}_x f_e, \tag{1.18}$$

$$h_{cex} f^{ey} = h_{ce}{}^y f_x{}^e. \tag{1.19}$$

The aggregate $(f_c{}^a, g_{cb}, f_c{}^x, f^a, f^x)$ satisfying (1.14) is said to be *normal* (*partially integrable*) if

(1.20) $$h_{ce}{}^x f_b{}^e + h_{be}{}^x f_c{}^e = 0,$$

(1.21) $$f_c{}^e \nabla_b f_b{}^x - f_b{}^e \nabla_e f_c{}^x - (\nabla_c f_b{}^e - \nabla_b f_c{}^e) f_e{}^x - (\nabla_c f_b - \nabla_b f_c) f^x = 0$$

holds respectively ([4]).

2. Tangential generic submanifolds of an odd-dimensional sphere

From now on we consider the generic submanifold M of $S^{2m+1}(1)$ such that the Sasakian ctructure vector V^h given by (1.13) is tangent to M, that is, $f^x=0$. Such a submanifold will be called a *tangential generic submanifold* of $S^{2m+1}(1)$.

We first prove

LEMMA 1. *Let M be a tangential generic submanifold with flat normal connection of an odd-dimensional unit sphere $S^{2m+1}(1)$. If the Ricci curvature of M is covariantly constant, then the induced structure on M is normal.*

Proof. Transvecting (1.9) with f^c and using (1.18) with $f^x=0$, we find

(2.1) $$K_{be} f^e = (n-1) f_b + h_{be}{}^x f_x{}^e - h_x f_b{}^x.$$

Differentiating (2.1) covariantly along M and substituting (1.16) and (1.17) with $f^x=0$, we get

(2.2) $$K_{be} f_c{}^e = (n-1) f_{cb} + (\nabla_c h_{be}{}^x) f_x{}^e + h_{be}{}^x h_{cax} f^{ea} - (\nabla_c h_x) f_b{}^x - h_x h_{ce}{}^x f_b{}^e$$

because the Ricci tensor of M is parallel.

If we take the skew-symmetric part of this and make use of (1.6) and (1.9), then we obtain

(2.3) $$(\nabla_c h_x) f_b{}^x - (\nabla_b h_x) f_c{}^x - 2 h_{be}{}^x h_{cax} f^{ea} - h_{ba}{}^x h_e{}^a{}_x f_c{}^e + h_{ca}{}^x h_e{}^a{}_x f_b{}^e = 0.$$

Transvecting (2.3) with $f_d{}^c f^b$ and taking account of (1.14) and (1.18) with $f^x=0$, we should have

(2.4) $$h_{de}{}^x f_x{}^e = P_x f_d{}^x - p f_d,$$

where we have put $P_x = h_{dey} f_x{}^d f^{ey}$.

Differentiating (2.4) covariantly and using (1.16) and (1.17) with $f^x=0$, we find

$$(\nabla_c h_{be}{}^x) f_x{}^e + h_{be}{}^x h_{cax} f^{ea} = (\nabla_c P_x) f_b{}^x + P_x h_{ce}{}^x f_b{}^e - p f_{cb}.$$

Thus, (2.2) reduces to

(2.5) $$K_{be} f_c{}^e = (n-p-1) f_{cb} + (\nabla_c P_x - \nabla_c h_x) f_b{}^x + (P_x - h_x) h_{ce}{}^x f_b{}^e.$$

Transvecting (1.9) with $f_y{}^b$ gives

(2.6) $$K_{ce} f_y{}^e = (n-p-1) f_{cy} + (h_x - P_x) h_{ce}{}^x f_y{}^e$$

with the aid of (1.8), (1.18) with $f^x=0$, (1.19) and (2.4).

If we transvect (2.5) with f^{cb} and take account of (1.14) and (1.18) with $f^x=0$, (2.1) and (2.6), we get

(2.7) $$K=n(n-p-1)-h_x(p^x-h^x).$$

Comparing this with (1.10), we obtain

(2.8) $$h_{cb}{}^x h^{cb}{}_x=h_x p^x+np.$$

Now, computing the length of square of $\nabla_c f_b{}^x$, we have

(2.9) $$\|\nabla_c f_b{}^x\|^2=h_{cb}{}^x h^{cb}{}_x-h_{ce}{}^x h_a{}^c{}_x f_y{}^e f^{ay}-p$$

with the aid of (1.4), (1.16), (1.18) and the fact that $f^x=0$.

Since the normal connection on M is flat, using the Ricci identity for $f_b{}^x$, we have

$$\nabla_d\nabla_c f_b{}^x-\nabla_c\nabla_d f_b{}^x=-K_{dcb}{}^a f_a{}^x,$$

which implies

$$(\nabla^b\nabla_c f_b{}^x)f_x{}^c=K_{cb}f^{bx}f_x{}^c$$

because of (1.16) with $f^x=0$. Thus, it follows that

(2.10) $$(\nabla^b\nabla_c f_b{}^x)f_x{}^c=(n-1)p+h_x p^x-h_{ce}{}^x h_b{}^e{}_x f_y{}^c f^{by},$$

where we have used (1.9).

On the other hand, we see from (1.14) and (1.16) with $f^x=0$ and (2.4) that $f_x{}^c\nabla_c f_b{}^x=0$. Applying ∇^b to this and substituting (2.10), we find

(2.11) $$(\nabla_c f_b{}^x)\ (\nabla^b f_x{}^c)=h_{ce}{}^x h_b{}^e{}_x f_y{}^c f^{by}-h_x p^x-(n-1)p.$$

Substituting (2.9) and (2.11) into the the identity

$$\frac{1}{2}\|\nabla_c f_b{}^x+\nabla_b f_c{}^x\|^2=\|\nabla_c f_b{}^x\|^2+(\nabla_c f_b{}^x)(\nabla^c f_x{}^b)$$

and taking account of (2.8), we should have

$$\nabla_c f_b{}^x+\nabla_b f_c{}^x=0.$$

Thus, (1.20) holds because of (1.16). This complete the proof of the lemma.

According to Theorem A and Lemma 1, we conclude

THEOREM 2. *Let M be an n-dimensional complete and tangential generic submanifold of an odd-dimensional unit sphere S^{2m+1} (1). If the normal connection is flat, the mean curvature vector of M is parallel in the normal bundle and the Ricci curvature of M is covariantly constant, then M is a pythagorean product of the form*

$$S^{p_1}(r_1)x\ldots\ldots xS^{p_N}(r_N),$$

where p_1, ..., p_N are odd numbers $\geqq 1$, $r_1{}^2+\ldots+r_N{}^2=1$, $N=2m+1-n$.

Transvecting (1.20) with $f_y{}^c f_a{}^b$ and making use of (1.14) and (1.18) with $f^x=0$, we find

(2.12) $$h_{ce}{}^{x}f_{y}{}^{e}=P_{yz}{}^{x}f_{c}{}^{z}-\delta_{y}{}^{x}f_{c},$$

where we have put $P_{yz}{}^{x}=h_{cb}{}^{x}f_{y}{}^{c}f_{z}{}^{b}$.

Substituting (2.12) into (1.12), then we see that the induced structure on M is partially integrable.

In particular, if M is proper Einstein space, then we have from (1.9)

$$h_{ce}{}^{x}h_{b}{}^{e}{}_{x}-h_{x}h_{cb}{}^{x}=\{n(n-1)-K\}/ng_{cb}$$

Transvecting this with $f_{y}{}^{c}f^{b}$ and taking account of (1.14) and (1.18) with $f^{x}=0$ and (2.4), we find $h_{y}=p_{y}$. Hence (2.7) becomes $K=n(n-p-1)$. Thus, it follows that $K-n(n-1)\neq 0$.

Owing to Theorem B, we have

THEOREM 3. *Let M be an n-dimensional complete and tangential generic submanifold of an odd-dimensional sphere $S^{2m+1}(1)$ with flat normal connection. If M is proper Einstein, then M is of the form*

$$S^{m}(r)\times S^{m}(r),\quad S^{q}(r)\times\ldots\ldots\times S^{q}(r)\ (N\text{-times}),\quad N_{q}=n$$

where q is an odd number and $2m-n+2=N$, $(N\neq n+2)$.

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