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ON A CLASS OF STARLIKE FUNCTIONS II

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1. Introduction

Let S denote the class of functions

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n$$

be analytic in the unit disk $U=\{|z|<1\}$. A function $f(z)\in S$ is said to be starlike of order $\alpha(0\leq\alpha<1)$ in the unit disk U if

$$\text{(1)}\qquad \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha$$

for all $z\in U$. And the above condition (1) is equivalent to

$$\left|\frac{zf'(z)/f(z)-1}{2\{zf'(z)/f(z)-\alpha\}-\{zf'(z)/f(z)-1\}}\right|<1.$$

O. P. Juneja and M. L. Mogra [3] studied the class of starlike functions

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n$$

in the unit disk U satisfying the condition

$$\text{(2)}\qquad \left|\frac{zf'(z)/f(z)-1}{2\beta\{zf'(z)/f(z)-\alpha\}-\{zf'(z)/f(z)-1\}}\right|<1$$

for some $\alpha(0\leq\alpha<1)$, $\beta(0<\beta\leq1)$ and $z\in U$. Recently S. Owa [6] showed some results for the class of starlike functions

$$f(z)=a_1z-\sum_{n=2}^{\infty}a_nz^n \qquad (a_n\geq0)$$

in the unit disk U satisfying the condition (2). Furthermore S. Owa [7] studied the class of functions

$$f(z)=a_1z-\sum_{n=2}^{\infty}a_nz^n \qquad (a_n\geq0)$$

analytic and starlike in the unit disk U satisfying the condition

$$\text{(3)}\qquad \left|\frac{zf'(z)/f(z)-1}{2\gamma\{zf'(z)/f(z)-\alpha\}-\{zf'(z)/f(z)-1\}}\right|<\beta$$

for $\alpha(0\leq\alpha<1)$, $\beta(0<\beta\leq1)$, $\gamma(0<\gamma\leq1)$ and $z\in U$.

In this paper, we consider about the class $S^*(\alpha,\beta,\gamma)$ of functions

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n$$

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analytic and starlike in the unit disk U satisfying the condition (3) for α $(0\leq\alpha<1)$, $\beta(0<\beta\leq 1)$, $\gamma(0<\gamma\leq 1)$ and $z\in U$. The classes $S^*(\alpha, 1, 1/2)$, $S^*\{0, 1, (2\gamma-1)/2\gamma\}$ $(\gamma>1/2)$, $S^*\{(1-\gamma)/(1+\gamma), 1, (1+\gamma)/2\}$ $(0<\gamma\leqq 1)$ and $S^*(1-\alpha, 1, 1/2)$ were studied by C. P. McCarty [5], R. Singh [9], [10], K. S. Padmanabhan [8] and P. J. Eenigenburg [2].

2. A representation formula

In the first place, we require the following lemma.

LEMMA 1. *Let a function*

$$H(z)=1+b_1z+b_2z^2+\dots$$

be analytic in the unit disk U. Then $H(z)$ is analytic and satisfies the condition

$$\left|\frac{H(z)-1}{2\gamma\{H(z)-\alpha\}-\{H(z)-1\}}\right|<\beta$$

for some $\alpha(0\leq\alpha<1)$, $\beta(0<\beta\leq 1)$, $\gamma(0<\gamma\leq 1)$ and all $z\in U$ if, and only if, there exists an analytic function $\phi(z)$ in the unit disk U such that $|\phi(z)|\leqq\beta$ for $z\in U$ and

$$H(z)=\frac{1+(2\alpha\gamma-1)z\phi(z)}{1+(2\gamma-1)z\phi(z)}$$

Proof. We employ the technique used by K. S. Padmanabhan [8]. Assume that a function

$$H(z)=1+b_1z+b_2z^2+\cdots$$

satisfies the condition

$$\left|\frac{H(z)-1}{2\gamma\{H(z)-\alpha\}-\{H(z)-1\}}\right|<\beta$$

for $\alpha(0\leq\alpha<1)$, $\beta(0<\beta\leq 1)$ and $\gamma(0<\gamma\leq 1)$. Setting

$$h(z)=\frac{1-H(z)}{2\gamma\{H(z)-\alpha\}-\{H(z)-1\}},$$

we see that the function $h(z)$ is analytic in the unit disk U, satisfies $|h(z)|<\beta$ for $z\in U$ and $h(0)=0$. Hence, by using Schwarz's lemma, we get $h(z)=z\phi(z)$, where $\phi(z)$ is an analytic function in the unit disk U and satisfies $|\phi(z)|\leq\beta$ for $z\in U$. Thus we obtain

$$H(z)=\frac{1+(2\alpha\gamma-1)h(z)}{1+(2\gamma-1)h(z)}=\frac{1+(2\alpha\gamma-1)z\phi(z)}{1+(2\gamma-1)z\phi(z)}.$$

On the other hand, if

$$H(z)=\frac{1+(2\alpha\gamma-1)z\phi(z)}{1+(2\gamma-1)z\phi(z)}$$

and $|\phi(z)|\leq\beta$ for $z\in U$, then $H(z)$ is an analytic function in the unit disk U. Furthermore, since $|z\phi(z)|\leq\beta|z|<\beta$ for $z\in U$, we get

$$\left|\frac{H(z)-1}{2\gamma\{H(z)-\alpha\}-\{H(z)-1\}}\right|=|z\phi(z)|<\beta$$

for $z\in U$. This completes the proof of the lemma.

THEOREM 1. *Let a function*

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n$$

be analytic in the unit disk U. *Then the function* $f(z)$ *is in the class* $S^*(\alpha, \beta, \gamma)$ *if, and only if,*

$$f(z)=z\ \exp\left\{-2\gamma(1-\alpha)\int_0^z\frac{\phi(t)}{1+(2\gamma-1)t\phi(t)}dt\right\}, \tag{4}$$

where $\phi(z)$ *is an analytic function in the unit* disk U *and satisfies* $|\phi(z)|\leq\beta$ *or* $z\in U$.

Proof. Let a function

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n$$

be in the class $S^*(\alpha, \beta, \gamma)$. Then, since the function $f(z)$ satisfies the condition (3), we can write

$$\frac{zf'(z)}{f(z)}=\frac{1+(2\alpha\gamma-1)z\phi(z)}{1+(2\gamma-1)z\phi(z)}$$

with the aid of Lemma 1. Consequently we obtain

$$\frac{f'(z)}{f(z)}-\frac{1}{z}=-\frac{2\gamma(1-\alpha)\phi(z)}{1+(2\gamma-1)z\phi(z)}.$$

On integrating both sides of the above equality from 0 to z, we have the representation formula (4).

Conversly, if $f(z)$ has the representation (4), it follows that

$$\frac{zf'(z)}{f(z)}=\frac{1+(2\alpha\gamma-1)z\phi(z)}{1+(2\gamma-1)z\phi(z)}$$

holds with $\phi(z)$ as in Lemma 1. Accordingly we have that $f(z)$ belongs to the class $S^*(\alpha, \beta, \gamma)$ with the aid of Lemma 1.

3. A distortion theorem

LEMMA 2. *Let a function*

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n$$

be in the class $S^*(\alpha, \beta, \gamma)$. *Then we have*

$$\frac{1+(2\alpha\gamma-1)\beta|z|}{1+(2\gamma-1)\beta|z|}\leq\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}\leq\frac{1-(2\alpha\gamma-1)\beta|z|}{1-(2\gamma-1)\beta|z|}$$

for $z\in U$.

Proof. Let a function

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n$$

belong to the class $S^*(\alpha, \beta, \gamma)$. Then, with an application of Schwarz's lemma, the condition (3) implies that $zf'(z)/f(z)$ assumes values lying in the disk obtained by taking the line segment joining two points $\{1+(2\alpha-1)\beta|z|\}/\{1+(2\gamma-1)\beta|z|\}$ and $\{1-(2\alpha\gamma-1)\beta|z|\}/\{1-2\gamma-1)\beta|z|\}$ as diameter. Hence we obtain

$$\frac{1+(2\alpha\gamma-1)\beta|z|}{1+(2\gamma-1)\beta|z|}\leq\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}\leq\frac{1-(2\alpha\gamma-1)\beta|z|}{1-(2\gamma-1)\beta|z|}.$$

THEOREM 2. *Let a function*

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n$$

be analytic in the unit disk *R and suppose* $f(z)\in S^*(\alpha, \beta, \gamma)$. *Then we have*

$$|f(z)|\geq\frac{|z|}{\{1+(2\gamma-1)\beta|z|\}^{2\gamma(1-\alpha)/(2\gamma-1)}}$$

and

$$|f(z)|\leq\frac{|z|}{\{1-(2\gamma-1)\beta|z|\}^{2\gamma(1-\alpha)/(2\gamma-1)}}$$

for $0\leq\alpha<1$, $0<\beta\leq1$, $0<\gamma\leq1$ *and* $\gamma\neq1/2$. *Further*

$$|z|\ \exp\{\beta(\alpha-1)|z|\}\leq|f(z)|\leq|z|\ \exp\{\beta(1-\alpha)|z|\}$$

for $0\leq\alpha<1$, $0<\beta\leq1$ *and* $\gamma\neq1/2$.

Proof. Since the function $f(z)$ is in the class $S^*(\alpha, \beta, \gamma)$, we have

$$\frac{zf'(z)}{f(z)}=\frac{1+(2\alpha\gamma-1)z\phi(z)}{1+(2\gamma-1)z\phi(z)},$$

where $\phi(z)$ is an analytic function in the unit disk U and $|\phi(z)|\leq\beta$ for $z\in U$. Therefore we obtain

$$\frac{f'(z)}{f(z)}-\frac{1}{z}=-\frac{2\gamma(1-\alpha)\phi(z)}{1+(2\gamma-1)z\phi(z)}. \tag{5}$$

On integrating both sides of (5) from 0 to z and taking real part of both sides of the resulting equation,

$$\begin{aligned}\log\left|\frac{f(z)}{z}\right|&=\operatorname{Re}\left\{\log\left(\frac{f(z)}{z}\right)\right\}=\operatorname{Re}\int_0^z\left\{\frac{f'(t)}{f(t)}-\frac{1}{t}\right\}dt\\&=\operatorname{Re}\int_0^z\left\{\frac{-2\gamma(1-\alpha)\phi(t)}{1+(2\gamma-1)t\phi(t)}dt\right.\\&\leq\int_0^{|z|}\frac{2\gamma(1-\alpha)|\phi(te^{i\theta})|}{|1+(2\gamma-1)te^{i\theta}\phi(te^{i\theta})|}dt.\end{aligned}$$

Consequently

$$\log\left|\frac{f(z)}{z}\right| \leq \int_0^{|z|} \frac{2\beta\gamma(1-\alpha)}{1-(2\gamma-1)\beta t} dt$$
$$= -\frac{2\gamma(1-\alpha)}{2\gamma-1} \log\{1-(2\gamma-1)\beta|z|\}$$
$$= -\log\{1-(2\gamma-1)\beta|z|\}^{2\gamma(1-\alpha)/(2\gamma-1)}$$

for $0\leq\alpha<1$, $0<\beta\leq 1$, $0<\gamma\leq 1$ and $\gamma\neq 1/2$. Moreover

$$\log\left|\frac{f(z)}{z}\right| \leq \beta(1-\alpha)\int_0^{|z|} dt = \beta(1-\alpha)|z|$$

for $0\leq\alpha<1$, $0<\beta\leq 1$ and $\gamma=1/2$. Hence we see that

$$|f(z)| \leq \frac{|z|}{\{1-(2\gamma-1)\beta|z|\}^{2\gamma(1-\alpha)/(2\gamma-1)}}$$

for $0\leq\alpha<1$, $0<\beta\leq 1$, $0<\gamma\leq 1$ and $\gamma\neq 1/2$ and

$$|f(z)| \leq |z| \exp\{\beta(1-\alpha)|z|\}$$

for $0\leq\alpha<1, 0<\beta\leq 1$ and $\gamma=1/2$.

On the other hand, by Lemma 2, we have

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geq \frac{1+(2\alpha\gamma-1)\beta|z|}{1+(2\gamma-1)\beta|z|}$$

for $0\leq\alpha<1$, $0<\beta\leq 1$, $0<\gamma\leq 1$ and $z\in U$. This gives

$$r\,\mathrm{Re}\left\{\frac{\partial}{\partial r}\left(\log\frac{f(z)}{z}\right)\right\} = \mathrm{Re}\left\{\frac{zf'(z)}{f(z)}-1\right\}$$
$$\geq \frac{1+(2\alpha\gamma-1)\beta r}{1+(2\gamma-1)\beta r} - 1 = -\frac{2\beta\gamma(1-\alpha)r}{1+(2\gamma-1)\beta r}$$

for $|z|=r$. Thus we see that

$$\log\left|\frac{f(z)}{z}\right| = \mathrm{Re}\left\{\log\frac{f(z)}{z}\right\} \geq \int_0^r \frac{-2\beta\gamma(1-\alpha)}{1+(2\gamma-1)\beta t} dt.$$

Hence

$$\log\left|\frac{f(z)}{z}\right| \geq -\frac{2\gamma(1-\alpha)}{2\gamma-1}\log\{1+(2\gamma-1)\beta r\}$$
$$= -\log\{1+(2\gamma-1)\beta\gamma\}^{2\gamma(1-\alpha)/(2\gamma-1)}$$

for $0\leq\alpha<1$, $0<\beta\leq 1$, $0<\gamma\leq 1$ and $\gamma\neq 1/2$ and

$$\log\left|\frac{f(z)}{z}\right| \geq \beta(\alpha-1)r$$

for $0\leq\alpha<1$, $0<\beta\leq 1$ and $\gamma=1/2$. Consequently we obtain

$$|f(z)| \geq \frac{|z|}{\{1+(2\gamma-1)\beta|z|\}^{2\gamma(1-\alpha)/(2\gamma-1)}}$$

for $0\leq\alpha<1$, $0<\beta\leq 1$, $0<\gamma\leq 1$ and $\gamma\neq 1/2$ and

$$|f(z)| \leq |z| \exp\{\beta(\alpha-1)|z|\}$$

for $0\leq\alpha<1$, $0<\beta\leq 1$ and $\gamma=1/2$. For equality we may take

$$f(z) = \frac{z}{\{1-(2\gamma-1)\beta z\}^{2\gamma(1-\alpha)/(2\gamma-1)}}$$

for $0\leq\alpha<1$, $0<\beta\leq 1$, $0<\gamma\leq 1$ and $\gamma\neq 1/2$ and

$$f(z)=z\ \exp\{\beta(1-\alpha)z\}$$

for $0\leq\alpha<1$, $0<\beta\leq 1$ and $\gamma=1/2$.

4. A sufficient condition for the class $S^*(\alpha, \beta, \gamma)$

THEOREM 3. *Let a function*

$$f(z)=z+\sum_{n=2}^{\infty} a_n z^n$$

be analytic in the unit disk U. If we have

$$\sum_{n=2}^{\infty}\{(n-1)+\beta(n+1-2\gamma n-2\alpha\gamma)\}|a_n|\leqq 2\beta\gamma(1-\alpha) \tag{6}$$

for some $0\leq\alpha<1$, $0<\beta\leq 1$ *and* $0<\gamma\leq 1/2$, *then the function* $f(z)$ *belongs to the class* $S^*(\alpha, \beta, \gamma)$.

Proof. We use a method of J. Clunie and F. R. Keogh [1]. Assume that the condition (6) holds. Then we get

$$\begin{aligned}
&|zf'(z)-f(z)|-\beta|2\gamma\{zf'(z)-\alpha f(z)\}-\{zf'(z)-f(z)\}|\\
&=\left|\sum_{n=2}^{\infty}(n-1)a_n z^n\right|-\beta\left|2\gamma(1-\alpha)z+\sum_{n=2}^{\infty}(1-2\alpha\gamma)a_n z^n-\sum_{n=2}^{\infty}(1-2\gamma)na_n z^n\right|\\
&\quad\leq\sum_{n=2}^{\infty}(n-1)|a_n||z|^n-\left\{\left|\,|2\beta\gamma(1-\alpha)z+\sum_{n=2}^{\infty}(1-2\alpha\gamma)\beta a_n z^n\right|\right.\\
&\qquad\qquad\left.-\sum_{n=2}^{\infty}(1-2\gamma)\beta n|a_n||z|^n\right\}\\
&\quad\leq\sum_{n=2}^{\infty}(n-1)|a_n||z|^n-\left\{2\beta\gamma(1-\alpha)|z|-\sum_{n=2}^{\infty}(1-2\alpha\gamma)\beta|a_n||z|^n\right.\\
&\qquad\qquad\left.-\sum_{n=2}^{\infty}(1-2\gamma)\beta n|a_n||z|^n\right\}\\
&\quad\leq\left[\sum_{n=2}^{\infty}\{(n-1)+\beta(n+1-2\gamma n-2\alpha\gamma)\}|a_n|-2\beta\gamma(1-\alpha)\right]|z|\\
&\quad\leq 0
\end{aligned}$$

for $0\leq\alpha<1$, $0<\beta\leq 1$, $0<\gamma\leq 1/2$ and $z\in U$. Consequently, by the maximum modulus theorem, the function $f(z)$ belongs to the class $S^*(\alpha, \beta, \gamma)$.

5. The radius of convexity for functions in the class $S^*(\alpha, \beta, \gamma)$

THEOREM 4. *Let a function*

$$f(z)=z+\sum_{n=2}^{\infty} a_n z^n$$

be in the class $S^*(\alpha, \beta, \gamma)$ *with* $0\leq\alpha<1$, $0<\beta\leq 1$ *and* $0<\gamma\leq 1/2$. *Then the function* $f(z)$ *maps*

$$|z|<\frac{1+\gamma(1-\alpha)-\sqrt{\gamma(1-\alpha)\{2+\gamma(1-\alpha)\}}}{\beta}.$$

on to a convex domain if

$$\left[1+\gamma(1-\alpha)+\sqrt{\gamma(1-\alpha)\{2+\gamma(1-\alpha)\}}\right]\left[\gamma(1-\alpha)+\sqrt{2\gamma(1-\alpha)\{1+\gamma(1-\alpha)\}}\right]$$
$$\leq \beta\sqrt{\gamma(1-\alpha)\{2+\gamma(1-\alpha)\}}$$
$$\leq \sqrt{\gamma(1-\alpha)\{2+\gamma(1-\alpha)\}}$$

This result is sharp.

Proof. We employ the technique used by T. V. Lakshminarasimhan [4]. Since the function $f(z)$ is in the class $S^*(\alpha, \beta, \gamma)$, by using Theorem 1, we obtain

$$\frac{zf'(z)}{f(z)}=\frac{1+(2\alpha\gamma-1)z\phi(z)}{1+(2\gamma-1)z\phi(z)},$$

where $\phi(z)$ is an analytic function in the unit disk U and satisfies $|\phi(z)|\leq\beta$ for $z\in U$. On differentiating both sides of the above equality with respect to z logarithmically, we get

$$1+\frac{zf''(z)}{f'(z)}=\frac{1+(2\alpha\gamma-1)z\phi(z)}{1+(2\gamma-1)z\phi(z)}-\frac{2\gamma(1-\alpha)\{z\phi(z)+z^2\phi'(z)\}}{\{1+(2\gamma-1)z\phi(z)\}\{1+(2\alpha\gamma-1)z\phi(z)\}}.$$

Moreover we have

$$\left|\frac{\phi'(z)}{\beta}\right|\leq\frac{1-|\phi(z)/\beta|^2}{1-|z|^2} \qquad (7)$$

for the analytic function $\phi(z)$ in the unit disk U. Since

$$\operatorname{Re}\left\{\frac{1+(2\alpha\gamma-1)z\phi(z)}{1+(2\gamma-1)z\phi(z)}\right\}$$
$$=\frac{1+(2\alpha\gamma-1)(2\gamma-1)|z\phi(z)|^2+2(\alpha\gamma+\gamma-1)\operatorname{Re}\{z\phi(z)\}}{|1+(2\gamma-1)z\phi(z)|^2}$$
$$\geqq\frac{\{1+(2\alpha\gamma-1)|z\phi(z)|\}\{1+(2\gamma-1)|z\phi(z)|\}}{|1+(2\gamma-1)z\phi(z)|^2}\geqq\frac{1+(2\alpha\gamma-1)|z\phi(z)|}{1+(2\gamma-1)|z\phi(z)|}$$

and

$$\operatorname{Re}\left(\frac{z\phi(z)+z^2\phi'(z)}{\{1+(2\gamma-1)z\phi(z)\}\{1+(2\alpha\gamma-1)z\phi(z)\}}\right)$$
$$\leqq\frac{|z\phi(z)|+|z^2\phi'(z)|}{\{1+(2\gamma-1)|z\phi(z)|\}\{1+(2\alpha\gamma-1)|z\phi(z)|\}}$$
$$\leqq\frac{|z\phi(z)|+|z^2\phi'(z)|}{\{1+(2\gamma-1)|z\phi(z)|\}^2},$$

we obtain

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}\geqq\frac{1+(2\alpha\gamma-1)|z\phi(z)|}{1+(2\gamma-1)|z\phi(z)|}-\frac{2\gamma(1-\alpha)\{|z\phi(z)|+|z^2\phi'(z)|\}}{\{1+(2\gamma-1)|z\phi(z)|\}^2}$$

If we assume that

(8) $1+|z\phi(z)|^2-2\{1+\gamma(1-\alpha)\}|z\phi(z)|-2\gamma(1-\alpha)|z^2\phi'(z)|>0,$

then we have

$$\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>0.$$

Now, in virtue of (7), the condition (8) will be satisfied

$$1+|z\phi(z)|^2-2\{1+\gamma(1-\alpha)\}|z\phi(z)|-2\gamma(1-\alpha)|z|^2\frac{\beta-|\phi(z)|^2/\beta}{1-|z|^2}>0.$$

Setting $a=|z|$ and $t=|z\phi(z)|$, the above condition can be re-written as

$$(1-a^2)[1+t^2-2\{1+\gamma(1-\alpha)\}t]-2\gamma(1-\alpha)\left(\beta a^2-\frac{t^2}{\beta}\right)>0,$$

that is,

$$(9)\qquad t^2\left\{(1-a^2)+\frac{2\gamma(1-\alpha)}{\beta}\right\}-2t(1-a^2)\{1+\gamma(1-\alpha)\}$$
$$+1-a^2-2\beta\gamma(1-\alpha)a^2>0,$$

where $0<a<1$ and $0\leq t\leq \beta a$. If $G(t)$ denotes the left hand member of (9), then we see that

$$G'(t)=2t\left\{(1-a^2)+\frac{2\gamma(1-\alpha)}{\beta}\right\}-2(1-a^2)\{1+\gamma(1-\alpha)\}.$$

Hence get $G'(t)=0$ for

$$t=t_1=\frac{\beta(1-a^2)\{1+\gamma(1-\alpha)\}}{\beta(1-a^2)+2\gamma(1-\alpha)}.$$

Furthermore

$$G''(t)=2\left\{(1-a^2)+\frac{2\gamma(1-\alpha)}{\beta}\right\}>0,$$

for $0<a<1$. Now $t_1-\beta a$ is positive and negative with

$$\beta a^3-\{1+\gamma(1-\alpha)\}a^2-\{\beta+2\gamma(1-\alpha)\}a+1+\gamma(1-\alpha),$$

respectively. Let

$$E(a)=\beta a^3-\{1+\gamma(1-\alpha)\}a^2-\{\beta+2\gamma(1-\alpha)\}a+1+\gamma(1-\alpha)$$

and let a_0 be the positive root of $E(a)=0$ lying in the open interval $(0, 1)$. Then $E(a)$ is positive for $0<a<a_0$ and so $t_1>\beta a$. Consequently $G'(t)$ is negative for $0\leq t\leq\beta a$, $G(\beta a)<G(t)$ and the condition (9) is satisfied if $G(\beta a)>0$. This is equivalent to

$$\beta^2a^2(1-a^2)-2\beta a(1-a^2)\{1+\gamma(1-\alpha)\}+1-a^2>0,$$

that is,

$$(1-a^2)[\beta^2a^2-2\beta\{1+\gamma(1-\alpha)\}a+1]>0$$

which holds for

$$a<\frac{1+\gamma(1-\alpha)-\sqrt{\gamma(1-\alpha)\{2+\gamma(1-\alpha)\}}}{\beta}.$$

Moreover we can show that

$$a_0>\frac{1+\gamma(1-\alpha)-\sqrt{\gamma(1-\alpha)\{2+\gamma(1-\alpha)\}}}{\beta}$$

if

$$1+\gamma(1-\alpha)-\sqrt{\gamma(1-\alpha)\{2+\gamma(1-\alpha)\}}\leq\beta\leq 1.$$

The condition on β implies that

$$\frac{1+\gamma(1-\alpha)-\sqrt{\gamma(1-\alpha)\{2+\gamma(1-\alpha)\}}}{\beta}<1,$$

and so

$$a_1=\frac{1+\gamma(1-\alpha)-\sqrt{\gamma(1-\alpha)\{2+\gamma(1-\alpha)\}}}{\beta}<a_0$$

if $E(a_1)$ is positive. Further $E(a_1)>0$ is satisfied if

$$\begin{aligned}(10)\qquad &\sqrt{\gamma(1-\alpha)\{2+\gamma(1-\alpha)\}}\,\beta^2\\ &-2\gamma(1-\alpha)[1+\gamma(1-\alpha)-\sqrt{\gamma(1-\alpha)\{2+\gamma(1-\alpha)\}}]\beta\\ &-[1+\gamma(1-\alpha)-\sqrt{\gamma(1-\alpha)\{2+\gamma(1-\alpha)\}}]^2\\ &\times\sqrt{\gamma(1-\alpha)\{2+\gamma(1-\alpha)\}}>0\end{aligned}$$

which holds if

$$\beta>\frac{1+\gamma(1-\alpha)-\sqrt{\gamma(1-\alpha)\{2+\gamma(1-\alpha)\}}}{\sqrt{\gamma(1-\alpha)\{2+\gamma(1-\alpha)\}}}\times[\gamma(1-\alpha)+\sqrt{2\gamma(1-\alpha)\{1+\gamma(1-\alpha)\}}].$$

Let

$$C(\alpha,\gamma)=\frac{1+\gamma(1-\alpha)-\sqrt{\gamma(1-\alpha)\{2+\gamma(1-\alpha)\}}}{\sqrt{\gamma(1-\alpha)\{2+\gamma(1-\alpha)\}}}\times(\gamma(1-\alpha)+\sqrt{2\gamma(1-\alpha)\{1+\gamma(1-\alpha)\}}]$$

If $\beta=C(\alpha,\gamma)$, then we have

$$a_0=\frac{\sqrt{\gamma(1-\alpha)\{2+\gamma(1-\alpha)\}}}{\gamma(1-\alpha)+\sqrt{2\gamma(1-\alpha)\{1+\gamma(1-\alpha)\}}}=\frac{1+\gamma(1-\alpha)-\sqrt{\gamma(1-\alpha)\{2+\gamma(1-\alpha)\}}}{\beta}$$

This shows that

$$|z|<\frac{1+\gamma(1-\alpha)-\sqrt{\gamma(1-\alpha)\{2+\gamma(1-\alpha)\}}}{\beta}$$

is mapped on to a convex domain by $f(z)$ provided $C(\alpha,\gamma)\leq\beta\leq 1$. To see that the estimate is sharp we choose

$$f(z)=\frac{z}{\{1+(2\gamma-1)\beta z\}^{2\gamma(1-\alpha)/(2\gamma-1)}}$$

so that $f(z)\in S^*(\alpha,\beta,\gamma)$ while

$$1+\frac{zf''(z)}{f'(z)}=0$$

when $$z=\frac{1+\gamma(1-\alpha)-\sqrt{\gamma(1-\alpha)\{2+\gamma(1-\alpha)\}}}{\beta},$$

$0\leq\alpha<1$, $0<\beta\leq 1$, $0<\gamma<1/2$, so that $f(z)$ is not convex in any disk $|z|<R$ if R exceeds

$$\frac{1+\gamma(1-\alpha)-\sqrt{\gamma(1-\alpha)\{2+\gamma(1-\alpha)\}}}{\beta}.$$

Furthermore, for $0\leq\alpha<1$, $0<\beta\leq 1$ and $\gamma=1/2$, we ought to choose

$$f(z)=z\exp\{\beta(\alpha-1)z\}.$$

This completes the proof of the theorem.

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