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LIFTING *-MORPHISMS OF UHF ALGEBRAS

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§1. Introduction

1.1 Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space, $\mathcal{L}(\mathcal{H})$ the algebra of all bounded operators $\mathcal{H}$, $\mathcal{K}(\mathcal{H})$ the ideal of compact operators, $\mathcal{C}(\mathcal{H})$ the quotient algebra $\mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$, and π the canonical homomorphism of $\mathcal{L}(\mathcal{H})$ onto $\mathcal{C}(\mathcal{H})$. Let X be a compact Hausdorff space and $C(X)$ be the algebra of all complex-valued continuous functions on X. Brown, Douglas and Fillmore initiated in their pioneer work [2] the study of the group Ext X consisting of unitary equivalence classes of unital *-monomorphism $\tau : C(X) \to \mathcal{C}(\mathcal{H})$. The group Ext X has many interesting features. Recently the theory of Ext has been generalized to non-commutative C^*-algebras with remarkable successes by work of many mathematicians (see for example [1]).

1.2 Let $\mathcal{M}$ be a II_∞-factor acting on H. It is well-known that $\mathcal{M}$ possesses an ideal analogous to $K(H)$; namely, the norm-closed two sided ideal $\mathcal{K}(\mathcal{M})$ generated by all finite projections in $\mathcal{M}$. Let $\mathcal{C}(\mathcal{M})$ denote the quotient algebra and π the canonical homomorphism. Then one can consider unitary equivalence classes of unital *-monomorphism $\tau : C(X) \to \mathcal{C}(\mathcal{M})$ and can ask whether there exists a parallel theory in the context of II_∞-factors. Fillmore [5] and Cho [4] indeed succeeded in developing extension theory to relative to II_∞-factors. Then the obvious question would be: is there any fruitful theory of extension relative to a II_∞-factor for separable nuclear C^*-algebras? Arveson's observation and Choi-Effros lifting theorm of completely positive maps (see for example [1]) make it possible that the unitary equivalence classes of unital *-monomorphism $\tau : \mathcal{A} \to \mathcal{C}(\mathcal{M})$ for a separable nuclear C^*-algebra $\mathcal{A}$ forms an abelian group provided that one can have the Voiculescu's non-commutative Weyl-von Neumann theorem analogue in the context of II_∞-factors.

1.3 In this note, we examine lifting problem of unital *-monomorphism $\tau : \mathcal{A} \to \mathcal{C}(\mathcal{M})$. Similar lifting problem for the classical Calkin algebra was

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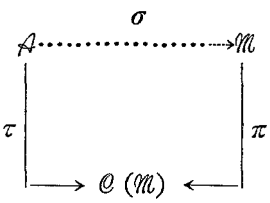
considered earlier by Thayer [6]. Our result says that for UHF algebras $\mathcal{A}$ any unital *-monomorphism $\tau : \mathcal{A} \to \mathcal{C}(\mathcal{M})$ can be lifted to unital *-monomorphism σ such that the following diagram commutes:



§ 2. Liftings for finite-dimensional C^*-algebra

2.1 Let $\mathcal{A}$ be a C^*-algebra. A family of partial isometries $\{e_{ij}\}^n_{i,j=1}$ of $\mathcal{A}$ such that

$$e_{ij}e_{km}=0 \text{ for } j \neq k$$
$$e_{ij}e_{km}=e_{im} \text{ for } j=k$$
$$e_{ij}{}^*=e_{ji}$$

is called a system of matrix units in $\mathcal{A}$.

For example if $\mathcal{A}$ is a full matrix algebra M_n, then the matrices e_{ij} having entries zero except in the (i, j)-position and having 1 in the (i, j)-position forms a matrix units in M_n. Let $\{u_{ij}\}^n_{i,j=1}$ be a matrix units in $\mathcal{C}(\mathcal{M})$. Then it is easy to see that any system of matrix units $\{e_{ij}\}$ in M_n determines a unique *-monomorphism $\tau : M_n \to \mathcal{C}(\mathcal{M})$ such that $\tau(e_{ij})=u_{ij}$ for all $i, j=1, 2, \ldots, n$. In order to find *-monomorphism $\sigma : M_n \to \mathcal{M}$ such that $\tau=\pi\circ\sigma$, we begin with the following lemma, which is type II_∞ version of Calkin's lifting theorem.

2.2 LEMMA. *Suppose that p and q are projections in $\mathcal{C}(\mathcal{M})$ such that $p \leq q$. Then there exist projections P and Q in $\mathcal{M}$ such that $P \leq Q$ and $\pi(P)=p$, $\pi(Q)=q$.*

Proof. First we will prove that for any projection q in $\mathcal{C}(\mathcal{M})$ there exists a projection Q in $\mathcal{M}$ with $\pi(Q)=q$. To prove this, we choose X in $\mathcal{M}$ such that $\pi(X)=q$. Since $\pi(X+X^*/2)=q$, we may and do assume that X is a self-adjoint element of $\mathcal{M}$. Let Q be the spectral projection of X corresponding to the interval $(\frac{1}{2}, \infty)$. Then the same arguement as in [2, Theorem 2.4] tells us that $X-Q$ is in the compact ideal $\mathcal{K}(\mathcal{M})$. Hence $\pi(Q)=q$. Since $p \leq q$, p belongs to $q\mathcal{C}(\mathcal{M})q$. But $\pi(Q\mathcal{M}Q)=q\mathcal{C}(\mathcal{M})q$ and hence there exists a p in $Q\mathcal{M}Q$ such that $\pi(P)=p$. This completes the proof.

2.3 LEMMA. *Suppose that P and Q are orthogonal projections in $\mathcal{M}$. Let u be a partial isometry in $\mathcal{C}(\mathcal{M})$ such that $u^*u=\pi(P)$, $uu^*=\pi(Q)$. Then there* *exists a partial isometry* U *in* $\mathcal{M}$ *with* $\pi(U)=u$, $U^*U\leq P$ *and* $UU^*\leq Q$.

Proof. Choose X in $\mathcal{M}$ with $\pi(X)=u$. Let $\pi(P)=p$ and $\pi(Q)=q$. Since $\pi(QXP)=q\pi(X)p=qup=u$, we may assume that X belongs to $Q\mathcal{M}P$. Let $X=V(X^*X)^{1/2}$ be its polar decomposition. Since $\pi((X^*X)^{1/2})=(u^*u)^{1/2}=\pi(P)$, by 2.2 there exists a projection E in $\mathcal{M}$ with $\pi(E)=\pi(P)$, namely the spectral projection of $(X^*X)^{1/2}$ corresponding to $(\frac{1}{2},\infty)$. Moreover E is a subprojection of the range projection of $(X^*X)^{1/2}$. Thus E is a subprojection of the initial projection of the partial isometry V. Hence VE is a partial isometry with the desired properties.

2.4 LEMMA. *Let* $\{e_{ij}\}^n_{i,j=1}$ *be a system of matrix units in* $\mathcal{C}(\mathcal{M})$. *Then there exists a system of matrix units* $\{E_{ij}\}^n_{i,j=1}$ *in* $\mathcal{M}$ *such that* $\pi(E_{ij})=e_{ij}$ *for all* i, $j=1,2,\ldots,n$.

Proof. We will prove this by the principle of mathematical induction. For $n=2$, by 2.2 we can choose two orthogonal projections F_{11} *and* F_{22} in $\mathcal{M}$ with $\pi(F_{11})=e_{11}$ and $\pi(F_{22})=e_{22}$. By 2.3 there exists a partial isometry E_{21} in $\mathcal{M}$ such that $\pi(E_{21})=e_{21}$, $E_{21}{}^*E_{21}\leq F_{11}$, $E_{21}E_{21}{}^*\leq F_{22}$. Since $F_{11}-E_{21}{}^*E_{21}$ and $F_{22}-E_{21}E_{21}{}^*$ are finite projections, we have $\pi(F_{11})=\pi(E_{21}{}^*E_{21})$ and $\pi(F_{22})=\pi(E_{21}E_{21}{}^*)$. Finally we put $E_{11}=E_{21}{}^*E_{21}$ and $E_{22}=E_{21}E_{21}{}^*$. Then $\{E_{ij}\}$, $1\leq i$, $j\leq 2$ is the desired system of matrix units. Now suppose that a system of matrix units $\{F_{ij}\}$, $2\leq i$, $j\leq n$, has been chosen so that $\pi(F_{ij})=e_{ij}$, $2\leq i$, $j\leq n$. Choose a projection F_{11} which is orthogonal to $F_{22}+F_{33}+\ldots+F_{nn}$ (possible by the proof of 2.2). Apply 2.3 to get a partial isometry E_{21} such that $\pi(E_{21})=e_{21}$, $E_{21}{}^*E_{21}\leq F_{11}$, $E_{21}E_{21}{}^*\leq F_{22}$. We put $E_{11}=E_{21}{}^*E_{21}$, $E_{22}=E_{21}E_{21}{}^*$. Then $\pi(E_{11})=\pi(F_{11})$ and $\pi(E_{22})=\pi(F_{22})$. We put $E_{i1}=F_{ii-1}\ldots F_{32}E_{21}$, for $i=3,4,\ldots,n$. Then $\{E_{i1}\}$, $2\leq i\leq n$, will generate the desired system of matrix units.

2.5 Let P be a finite projection in $\mathcal{L}(\mathcal{H})$. Let n be a natural number. Then there exist mutually orthogonal equivalent projections $P_1, P_2, \ldots, P_n$ such that $P_1+P_2+\ldots+P_n=P$ if and only if n divides the vector space dimension of the range space of P.

However, in a II_∞-factor $\mathcal{M}$ for any finite projection P and for any natural number n there exist mutually orthogonal equivalent projections $P_1, P_2, \ldots, P_n$ such that $P_1+P_2+\ldots+P_n=P$. This distiction makes the following theorem possible.

2.6 THEOREM. *For any unital *-monomorphism* $\tau: M_n\rightarrow\mathcal{C}(\mathcal{M})$, *there exists a unital *-monomorphism* $\sigma: M_n\rightarrow\mathcal{M}$ *such that* $\tau=\pi\circ\sigma$.

Proof. Let $\{e_{ij}\}^{n}_{i,j=1}$ be a system of matrix units for M_n. It suffices to show that there exists a system of matrix units $\{F_{ij}\}$ in $\mathcal{M}$ such that $\pi(F_{ij})=\tau(e_{ij})$ and $F_{11}+F_{22}+\ldots+F_{nn}=1$. Let E_{ij} be a system of matrix units in $\mathcal{M}$ which lifts $\tau(e_{ij})$ (it is possible by 2.4). Since $\pi(E_{11}+\ldots+E_{nn})=1$, the projection $P=1-(E_{11}+\ldots+E_{nn})$ is finite in $\mathcal{M}$. Choose mutually equivalent orthogonal projections $P_1, P_2, \ldots, P_n$ such that $P_1+P_2+\ldots+P_n=P$. Let U_{i1} be a partial isometry connecting P_1 to P_i. Note that P_i and U_{i1} are compact elements in $\mathcal{M}$. Set $F_{ii}=E_{ii}+P_i$ and $F_{i1}=E_{i1}+U_{i1}$. Then $\{F_{i1}\}_{2<i<n}$ will generate a system of matrix units $\{F_{ij}\}$ with $F_{11}+F_{22}+\ldots+F_{nn}=1$. This completes the proof.

Since any finite dimensional C^*-algebra is a direct sum of full matrix algebras, we get:

2.7 COROLLARY. *Let $\mathcal{A}$ be a finite dimensional C^*-algebra. Let $\tau : \mathcal{A} \to \mathcal{C}(\mathcal{M})$ be a unital $*$-monomorphism. Then there exists a unital $*$-monomorphism $\sigma : \mathcal{A} \to \mathcal{M}$ such that $\tau=\pi\circ\sigma$.*

2.8 LEMMA. *Let $\mathcal{A}$ be a full matrix algebra. Suppose that σ_1 and σ_2 are unital $*$-monomorphisms of $\mathcal{A}$ into $\mathcal{M}$. Then there exists a unitary U in $\mathcal{M}$ such that $\sigma_1(x)=U^*\sigma_2(x)U$ for all x in $\mathcal{A}$.*

Proof. Let $\{E_{ij}\}_{1\leq i,j\leq n}$ be a system of matrix units for $\mathcal{A}$. We put $e_{ij}=\sigma_1(E_{ij})$ and $f_{ij}=\sigma_2(E_{ij})$. It suffices to show that there exists a unitary U in $\mathcal{A}$ such that $e_{ij}=U^*f_{ij}U$ for all $1\leq i,\ j\leq n$. To this end, since $\mathcal{M}$ is a σ-finite II_∞-factor, the infinite projections e_{11} and f_{11} are equivalent. Hence there exists an element V of $\mathcal{M}$ such that $V^*V=e_{11}$ and $VV^*=f_{11}$. We put $\mathrm{U}=\sum_{i=1}^{n} f_{i1}Ve_{1i}$. Then it is easy to check that $e_{ij}=U^*f_{ij}U$. This completes the proof.

2.9 COROLLARY. *Let $\mathcal{A}$ be a finite dimensional C^*-algebra. Let $\tau_1, \tau_2 : \mathcal{A} \to \mathcal{C}(\mathcal{M})$ be unital $*$-monomorphisms. Then there exists a unitary U in $\mathcal{M}$ such that*

$$\tau_1(x)=\pi(U)^*\tau_2(x)\pi(U)$$

for all x in $\mathcal{A}$.

Proof. According to Corollary 2.7, each τ_i has a unital lifting σ_i satisfying $\tau_i=\pi\circ\sigma_i$, $i=1,2$. Application of Lemma 2.8 to summand by summand will give us a unitary U implementing the requirement.

2.10 REMARK. Two extensions (i.e., unital $*$-monomorphisms) $\tau_1, \tau_2 : \mathcal{A} \to \mathcal{C}(\mathcal{M})$ are said to *equivalent* if there exists a unitary U in $\mathcal{M}$ such that $\tau_1(x)=\pi(U)^*\tau_2(x)\pi(U)$ for all x in $\mathcal{A}$. The sum of $\tau_1+\tau_2$ is the extension $\tau : \mathcal{A} \to \mathcal{C}(\mathcal{M})$ defined as follows: choose isometries V_1 and V_2 in $\mathcal{M}$ such that $V_1V_1^*+V_2V_2^*=1$, and let

$$(\tau_1+\tau_2)(x)=\pi(V_1)\tau_1(x)\pi(V_1^*)+\pi(V_2)\tau_2(x)\pi(V_2^*)$$

for all x in $\mathcal{A}$. The equivalence class of $\tau_1+\tau_2$ is independent of the choice of isometries in the definition. Let $\mathrm{Ext}^{\mathcal{M}}\mathcal{A}$ denote the equivalence classes of extensions. Then for commutative C^*-algebra $\mathcal{A}$ $\mathrm{Ext}^{\mathcal{M}}\mathcal{A}$ is an abelian group (see [4] for details). Corollary 2.9 can be restated as follows: For finite dimensional C^*-algebra $\mathcal{A}$, the $\mathrm{Ext}^{\mathcal{M}}\mathcal{A}$ is always trivial group.

We close this section with the following.

2.11 THEOREM. *Suppose that $\mathcal{A}_1$ and $\mathcal{A}_2$ are full matrix algebras and that $\mathcal{A}_1$ is a subalgebra of $\mathcal{A}_2$ with the same unit. If $\tau_1, \tau_2 : \mathcal{A}_2 \to \mathcal{C}(\mathcal{M})$ is a unital*-monomorphism and $\sigma_1 : \mathcal{A}_1 \to \mathcal{M}$ is a unital *-monomorphism such that $\pi\circ\sigma_1=\tau_1$, and then $\tau_2/\mathcal{A}_1=\tau_1$ there exists a unital *-monomorphism $\sigma_2 : \mathcal{A}_2 \to \mathcal{M}$ such that $\pi\circ\sigma_2=\tau_2$ and $\sigma_2|\mathcal{A}_1=\sigma_1$.*

Proof. Let $\{e_{ij}\}^n_{i,j=1}$ be a system of matrix units for $\mathcal{A}_1$ and $\{f_{ij}\}^m_{i,j=1}$ be a system of matrix units for $\mathcal{A}_2$. Since $\mathcal{A}_1\subset\mathcal{A}_2$, n devides m. Let $m=kn$. By rearranging f_{ij} if necessary, we can assume that $e_{11}=f_{11}+f_{22}+\ldots+f_{kk}$, $e_{22}=f_{k+1k+1}+\ldots+f_{2k2k}$, ..., By applying 2.5 to $\sigma_1(e_{11})$, $\tau_2(f_{ii})$, $i=1, 2, \ldots, k$; $\tau_2(f_{i1})$, $i=2,\ldots,k$, we get a system of matrix units $\{F_{ij}\}_{1\le i,\ j\le k}$ in $\mathcal{M}$ such that

1) $F_{11}+F_{22}+\ldots+F_{kk}=\sigma_1(e_{11})$
2) $\pi(F_{ij})=\tau_2(f_{ij})$, $1\le i,\ j\le k$

For each $j=2,3,\ldots,n$, $i=2,3,\ldots,k$, we put

3) $F_{1+(j-1)k,1}=E_{j1}F_{11}$ and $F_{i+(j-1)k,1}=E_{j1}F_{i1}$

Then these partial isometries $\{F_{s1}\}_{2\le s\le m}$ together with properties (1) and (2) furnish us with the desired *-monomorphism $\sigma_2 : \mathcal{A}_2\to\mathcal{M}$

§3. Liftings for UHF algebras

A C^*-algebra $\mathcal{A}$ with unit is uniformly hyper-finite (UHF) if there is an increasing sequence $\{\mathcal{A}_n\}$ of full matrix subalgebras containing the same unit of $\mathcal{A}$ and such that $\overline{\bigcup_{n=1}^{\infty} \mathcal{A}_n}=\mathcal{A}$.

THEOREM. *Let $\mathcal{A}$ be a* UHF *algebra with $\overline{\bigcup\mathcal{A}_n}=\mathcal{A}$, where $\mathcal{A}_n$ is increasing sequence of full matrix subalgebras. Let $\tau : \mathcal{A}\to\mathcal{C}(\mathcal{M})$ be a unital *-monomorphism. Then there exists a unital *-monomorphism $\sigma : \mathcal{A}\to\mathcal{M}$ such that $\pi\circ\sigma=\tau$.*

Proof. Let σ_1 be a unital *-monomorphism of $\mathcal{A}_1$ into $\mathcal{M}$ such that $\pi\circ\sigma_1=\tau|\mathcal{A}_1$ (such a σ_1 exists by 2.6). By 2.11 we can extend σ_1 to a *-monomorphism σ_2 of $\mathcal{A}_2$ into $\mathfrak{M}$ such that $\pi\circ\sigma_2=\tau|\mathcal{A}_2$. Thus by keeping doing this process, we get a unital *-monomorphism $\bar{\sigma}:\overline{\cup\mathcal{A}_n}\to\mathfrak{M}$ such that $\pi\circ\sigma=\tau|\cup\mathcal{A}_n$. Let σ be the unique extension of $\bar{\sigma}$ to $\mathcal{A}$. Now it is clear that $\pi\circ\bar{\sigma}=\tau$. This completes the proof.

References

1. W.B. Arveson, *Notes on extensions of C*-algebras*, Duke Math. J. **44** (1977), 329-355.
2. L.G. Brown, R.G. Douglas and P.A. Fillmore, *Unitary equivalence modulo the compact operators and extensions of C*-algebras*, Lecture Notes in Mathematics, 345, Springer-Verlag, New York, 1973.
3. J.W. Calkin, *Two-sided ideals and congruences in the ring of bounded operators in Hilbert space*, Ann. of Math., **42** (1941), 839-872.
4. S.J. Cho, *Extensions relative to a* II_∞*-factor*, Proc. Amer. Math. Soc. **74** (1979), 109-112.
5. P.A. Fillmore, *Extensions relative to semi-finite factors*, Istituto Nazionale di Alta Mathematica, Simp. Math. **20** (1976), 486-496.
6. F.J. Thayer, *Obstructions to Lifting *-monomorphisms into the Calkin algebra*, [1]. J. of Math. **20** (1976) No. 2, 322-328.

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