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A Sense of Numeration History for the Elementary School Teacher

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The importance of the whole numbers (0, 1, 2, 3, 4, ...) to the field of mathematics can hardly be overstated. They are the building blocks of mathematics and thus a starting point for the study of mathematics. A substantial portion of the mathematics curriculum of the elementary school is devoted to developing an understanding of these numbers, their relative sizes, the four basic operations on them, algorithms for performing these operations, and how these numbers can be applied to solve real world problems. Does this list contain all the major whole number topics of the elementary school curriculum? Definitely not! ***High on any list of important whole number topics should be the development of an efficient system for naming the whole numbers.*** Without such a system, we would not only be severely curtailed in our efforts to record numbers, but we would also experience difficulty in determining the smaller or larger of two numbers, and in computing their sum, difference, product, or quotient. Fortunately, we have a marvelous numeration system for doing this, namely, the Decimal System (also referred to as the Base Ten or Hindu-Arabic System).

The spiral development of the decimal system is a major strand in the elementary school mathematics curriculum. Unfortunately, as many teachers and researchers acknowledge, this strand is beset with problems. More students than we care to admit have difficulty in learning the decimal system. A lack of understanding of this system hampers students in many ways. For example, many of the computational problems that students experience can be traced back to problems with numeration. Why so much trouble in learning the decimal system? Payne and Rathmell (4) hypothesize:

> Children often have difficulty with numeration because too many different but closely related tasks are presented at nearly the same time. It is not uncommon for children to be expected to count by ones, count by tens, group by tens, make tens and ones charts, name numbers orally, read and write two-digit numerals, recognize the significance of ten in our base-ten system of numeration, recognize the equivalence of numbers named orally and named by their respective base representations, and perhaps grasp several other ideas, all during their initial exposure to tens and ones. Children need time and assistance to sort out these base and place-value ideas and to relate all the components of numeration.

It is probably safe to say that most elementary school teachers experience little difficulty in *using* the decimal system for whole numbers. However, it is probably not safe to say that most of them have the depth of understanding that is desired for *teaching* this system. Their many years of using the system mechanically may have made it so familiar to them that they only see it as a whole, and are unaware of the components that comprise it. To reverse a wellknown chiche, they fail to see the trees for the forest. The decimal system contains several *layers of abstraction*. Thus, a cmplete understanding of it does not come all at once, but only from a careful sequencing of ideas that are developed through the elementary grades. An elementary school techer needs to have an understanding and appreciation of these layers of abstration which took centuries to evolve.

The stages of numeration development that eventually led to the discovery of the decimal system are indicative of the stages that many students need to experience to gain eventually a thorough understanding of it. *In this regard, a sence of numeration history can serve an elementary School teacher well.* Such a sense can provide him or her with a depth of understanding that will help in teaching the decimal system.

The main part of this paper will consist of a stage-by-stage creation of a hypothetical numeration system. This system is an example of a logically possible historical development, and so should provide the reader with a sense of numeration history. In its final stage of development, the system possess the basic characteristics of the decimal system. At the end of this article, the goals that you should have realized will include these three: (a) be able to identify the several layers of abstraction and the major principles that exist in the decimal system; (b) have a better understanding of the distinction between number and numeral and between zero as a placeholder and number; (c) be motivated to read on historical numeration systems (e.g., Egyptian, Greek, Mayan, Chinese) and have a framework available for placing and interpreting them.

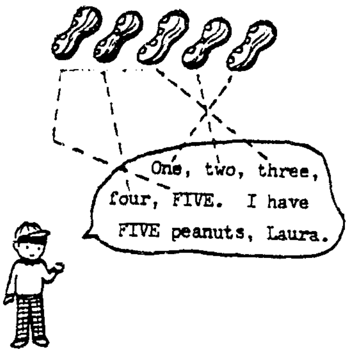
To help describe the stages in the development of this hypothetical system, the decimal system will be used, but only in a mechanical way. Our general interest in the construction of this hypothetical system is to gain a better understanding and appreciation of the decimal system.

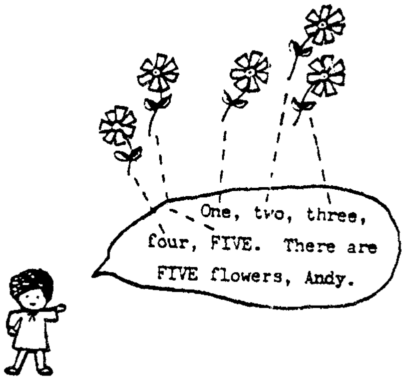
Since we want to construct a *whole number numeration system* (i.e., a set of basic symbols and a scheme for employing them to represent the whole numbers), it worthwhile to explore briefly the concept of a whole number. We begin by exploring the concept of a counting number (i.e., a non-zero whole number).

Counting Number Concept

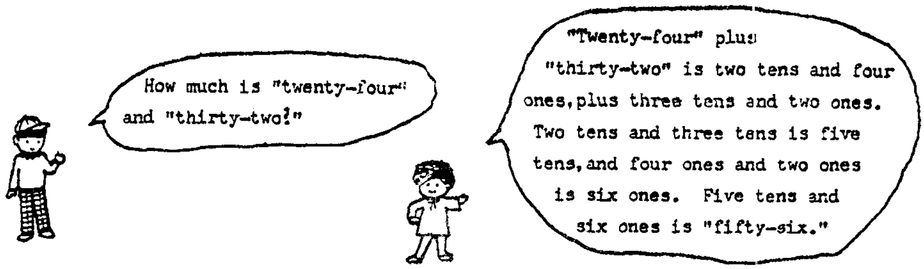
The concept of a counting number can be attained by relating several elementary concepts that underlie it, namely, sets, matching sets, and counting. For example, an activity leading to the concept of the counting number five is the repeated counting of the objects in matching sets, as illustrated.

The illustration shows that *counting the objects of a set (or deaterming the number of a set)* consists of matching part of the counting chant, beginning with "one" to objects of the set (in any order) with the purpose of using the last word uttered to represent the number of elements in the set. Thus, the counting number *five* can be viewed as the abstraction that results from counting the objects of sets that match, say, the set of peanuts or the set of flowers. The other counting numbers can be described similarly.





We have just seen that the counting number concept exists independent of *visual* symbols for the numbers, i.e., *numerals.* Many young children cannot write or even recognize decimal numerals, yet they have the counting number concept. Coupling this concept with their knowledge of *oral* numerals has resulted in many of them being proficient in performing meaningful and rather complex computations on these numbers mentally.

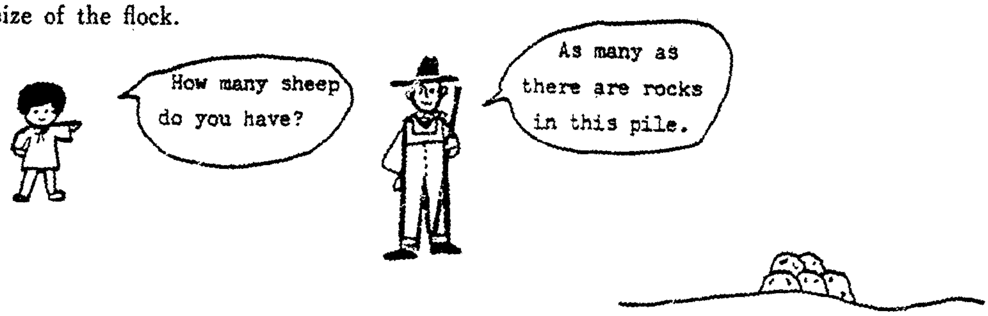


Of course, a child's progress in mathematics would be severely impeded if an efficient system of visual system of visual symbols for the counting numbers were not eventually made available to the child. But don't be too hasty in having your children work with decimal numerals. *Mental and and oral work in arithmetic, especially in the primary grades, pays many dividends later when students are working with decimal numerals.* Let us now go back in history, where we begin the construction of our hypothetical numeration system. (The remainder of the article may be more beneficial to you if you work the several problems that appear throughout.)

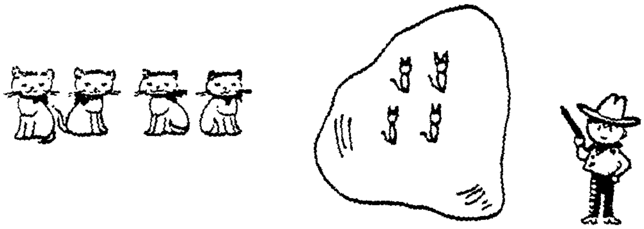
Hypothetical Numeration System

Stage 0. In the beginnings of this stage, the most primitive type of numerals, namely, *objects* such as sticks and stones, are used to keep track of the size of a set. The scheme for doing this is simple - match the set with a collection of sticks or stones as shown below for a flock of sheep.

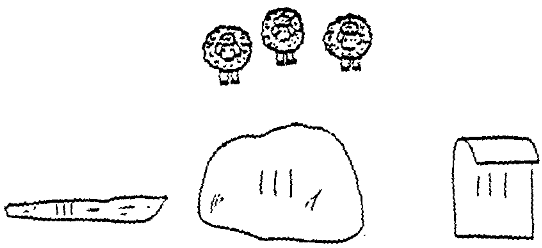
If these sheep go out to pasture for the day and someone is interested in the size of the flock, the shepherd can convey this information by using the above rocks. That is, the rocks are a *record* of the size of the flock.



At the end of this stage, we see the use of *pictures* to record the size of a set.



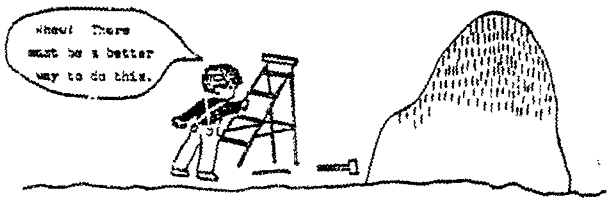
Stage 1. In the furthre efforts of primitive people to record the size of a set, there was a gradual progression from the use of objects or pictures, to *tally marks* made by a substance on papyrus. The use of these visual symbols for the size of a shepherd's flock of sheep is shown.



The scheme employed by this tally system is simple, indeed - write a tally mark for each of the objects in the collection. It is solidly based on the concept of one-to-one correspondence.

Unfortunately for the tally system, the need became greater for keeping permanent records of large sets. As an example, victorious empires wanted to keep track of their spoils, The deficiency of the tally system is perhaps best illustrated by this question: "How would you like to chisel in stone the numeral for the number of captured warriors, say, 20, 000?"

(The tally system lends itself to some very simple algorithms for adding, subtracting, multiplying, and dividing. Can you give an example of each?)

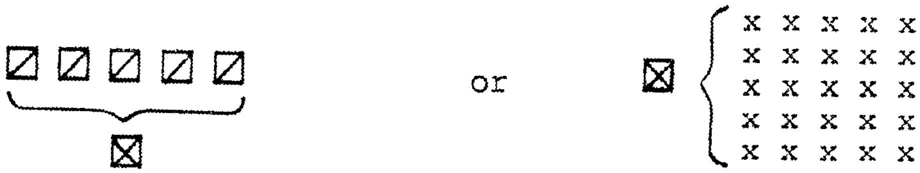


Stage 2. This stage can be viewed as the tally stage with abbreviations. The first four numerals are unchanged. The fifth numeral is abbreviated by using a ***grouping*** scheme: ||||| is replaced by 𝍸. The numerals following 𝍸 are 𝍸|, 𝍸||, 𝍸|||, and so on. This grouping process makes it easier to read a numeral. That is, it makes it easier to fathom the size of the number represented by, say, the numeral 𝍸 𝍸 𝍸 ||| as opposed to the numeral ||||||||||||||||||. Having a special numeral, in this case 𝍸, for a group of tally marks is a big step in the progress of numeration systems. (***Can you devise some computational algorithms for this stage?***)

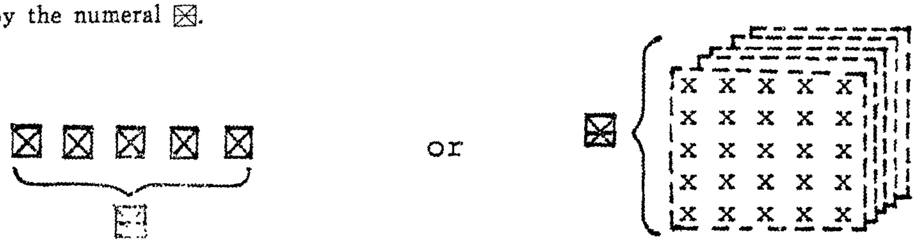
Even with this grouping feature, it is apparent that the system is too cumbersome. To economize on the amount of writing required to record a numeral, we progress to Stage 3.

Stage 3. In this stage, tally marks are again used to represent the numbers one through four, but the symbol ⧄ is the numeral for the number five (5).

Other symbols are introduced to represent further groupings of five. Five groups of ⧄ are represented by the numeral ⊠.



Thus, ⊠ is a numeral for the number we know as twenty-five (5^2). Five groups of ⊠ are represented by the numeral ⧆.



Thus, ⧆ is a numeral for the number we know as one-hundred twentyfive (5^3). Other basic symbols are then introduced for five groups of ⧆, and so on. For the ideas to be conveyed here, the basic numerals |, ||, |||, ||||, ⧄, ⊠, ⧆, suffice. Since a special numeral is introduced for each grouping of five, the *base* of the system is ⧄, or five. What we have introduced then, are ***special symbols for powers of the base***, that is, for $5^1, 5^2, 5^3$, and so on.

The scheme for employing these numerals to represent the counting numbers is illustrated, thusly:

1	2	3	4	5	6	7	8	9	10
|	||	|||	||||	⧄	⧄|	⧄||	⧄|||	⧄||||	⧄⧄

11	····	14	15	16	···	24	25	
⧄⧄|		⧄⧄||||	⧄⧄⧄	⧄⧄⧄|		⧄⧄⧄⧄||||	⊠	
26	····	31	···· 124	125	126	···	181···	
⊠|		⊠⧄|	⊠⊠⊠⊠⧄⧄⧄⧄||||	⧆	⧆|		⧆⊠⊠⧄|	

The scheme is easily described: the numeral for a number can be derived from the numeral for the preceding number by joining one tally mark to this numeral, and replacing (trading) any five groups of

|, ⧄, or ⊠

by

⧄, ⊠, or ⧆,

respectively. (*What is the Stage* 3 *numeral for* 312? ***For*** 624?)

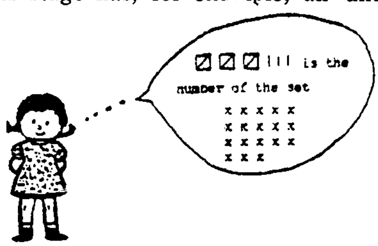
This system is an ***additive system*** because the number represented by a particular numeral is determined by adding the numbers represented by the basic numerals comprising the numeral.

⧄⧄⧄⧄|||=23, since 5+5+5+5+3=23.

⧆⊠⧄|=156 (*Do you see why*?)

(*What is the number represented by* ⊠⧄⧄? *By* ⧄⊠⧄?) It is easily seen that this system requires far less writing in the recording of numerals than did the system of Stage 2.

At this point in the development, it is important to emphasize again that a knowledge of the decimal system is not a prerequisite for understanding and working wit hthis or any other stage of our hypothetical system. It is just convenient to use the decimal system in describing these stages to you. Anyone working ***within*** this stage has, for example, an understanding of the number ⧄⧄⧄||| as follows:



The numerals for the numbers that are one less than the base are called the *digits* of the system. The digits for Stage 3 are |, ||, |||, and ||||. (*Can you devise some computational algorithms for this stage of numeration*?)

Stage 4. We now progress to a system that is still additive with a base, but also employs ***multiplicative*** and ***positional*** ideas.

Recall that the first four numerals (the digits) of the Stage 3 system were formed by repeating the numeral for one. In this stage, we do not want to repeat numerals in forming basic symbols; hence, we replace the the numerals

|| , |||, and ||||

by

∧, △, and □,

respectively. Thus the basic numerals in this stage, for the given numbers, are

1	2	3	4	5	25	125	625...
|	∧	△	□	⧄	⊠	⧆	and so no.

We now focus our attention on two sets of numerals, the digits

{|, ∧, △, □}

and the remaining numerals

A={⧄, ⊠, ⧆}.

To represent a number greater than four in this system, we will use pairs on numerals. A particular pair consists of a digit (written on the left) and a numeral from set A (written on the right). The digit of a pair represents the number of sets of the size indicated by the other numeral. For example, the pair ∧⧄ represents ⧄⧄. That is, the numeral ∧⧄ means two fives in decimal language. Other examples follow.

△⊠=⊠⊠⊠ □⧆=⧆⧆⧆⧆

A numeral may contain any number of these pairs; thus, the numeral

△⧆□⊠∧⧄|

is a shorthand expression for

⧆⧆⧆⊠⊠⊠⊠⧄⧄|.

Using decimal notation, we see that

△⧆□⊠∧⧄| $=3\cdot125+4\cdot25+2\cdot5+1=486$

(Can you find the numeral for 123? *For* 626?)

The example just presented indicates how this system employs additive and multiplicative principles. The numbers represented by the numerals in a pair are ***multiplied*** and then these products are ***added*** to determine the number represented by the numeral.

It is important to notice that △⊠∧⧄ and ∧⊠△⧄ are not equal, even though the same four symbols appear in the two numerals. We have

△⊠∧⧄ $=3\cdot25+2\cdot5=75+10=85$,

and

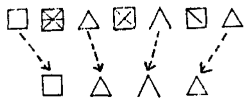
∧⊠△⧄ $=2\cdot25+3\cdot5=50+15=65$.

Consequently, this system, unlike the system of Stage 3, is ***positional.*** That is, a change in the position of a symbol in a numeral may change the number represented by the numeral. (Again, as in Stage 3, we have economized on the number of symbols used in a numeral.)

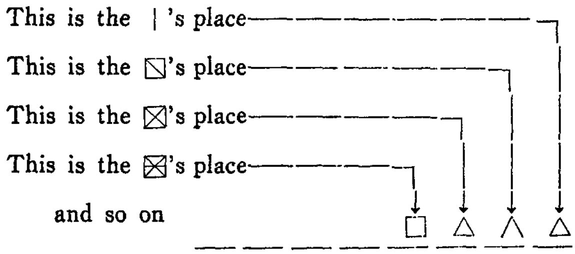
Stage 5. This is the last stage in the development of the hypothetical numeration system. The system developed here is additive, multiplicative, and positional, as is the system of Stage 4, but it also employs a ***place-value*** idea. It is obtained, quite simply, by eliminating the need for the symbols ⧄, ⊠, and ⧆.

For example, the Stage 4 numeral

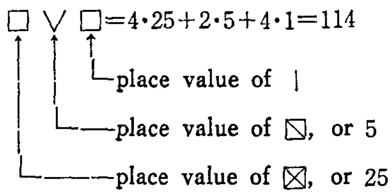
is now erpresented as



It is here that the place-value idea is used; that is, values (powers of the base) are assigned to specific places in a numeral (the first place from the right, the second place from the right, etc.) according to this scheme:



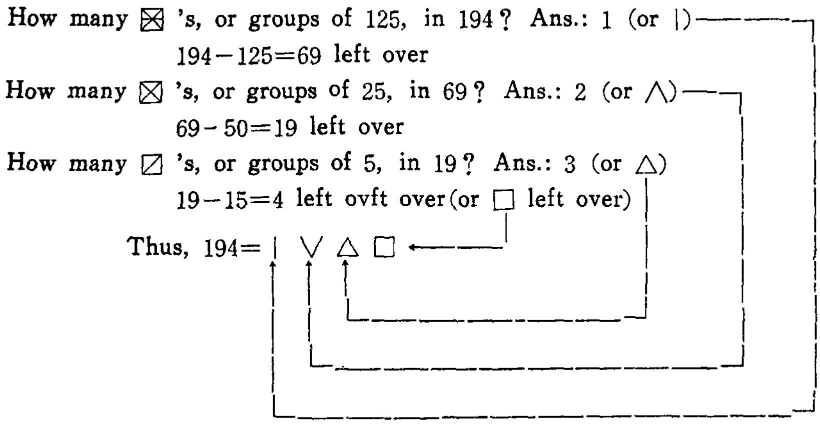
We notice that each place to the left of the one's place represents five times the value of the place to its immediate right. Once the values of the places are known, the place-value, additive, and multiplicative ideas can be used to determine the decimal numeral for a number having a Stage 5 numeral (now consisting only of digits).



(*Can you find decimal numeral for* ∧ | □ △ ?)

Conversely, the Stage 5 numeral 194 is found by finding the appropriate digits for each place: ___ ___ ___ ___

⊠ ⊠ ⧄ |



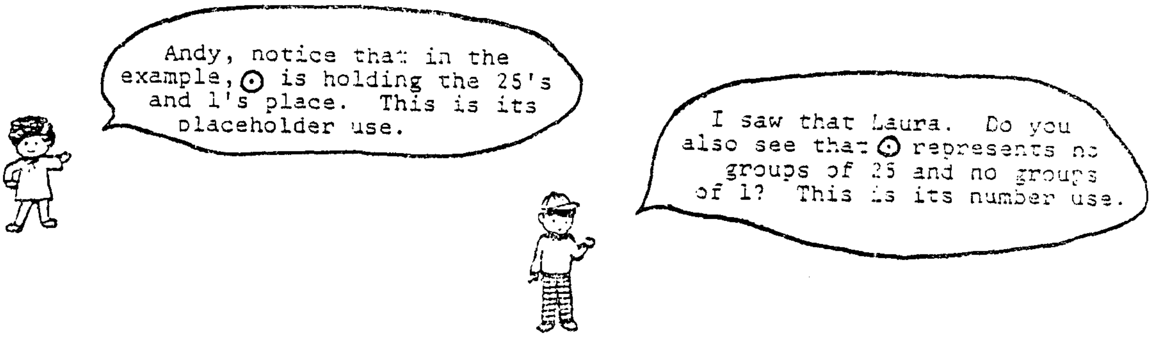
(*Can you find the Stage* 5 *numeral for* 493?)

The need for a placeholder (*Stage* 5 *continues.*) What is the Stage 5 numeral for 51? Since 51=

2·25+1, the Stage 4 numeral is ∧ ⊠ |. However, ∧| (obtained by writing in order the digits of this Stage 4 numeral) is equal to ∧ ⧅ |, or 2·5+1, which equals 11. The digit ∧ must occur in the *third* place from the right, not the second. There are at least two ways to handle this. One way is to leave a space (e.g., ∧ |), as the Babylonians did around 3,000 B.C.. A second is to invent a ***placeholder;*** that is, a symbol to take up the empty space. Since the first option is undesirable (*why?*), the second is chosen, so we introduce the symbol ⊙ as the placeholder. We now can write the Stage 5 numeral for 51, namely, ∧ ⊙ |. (*What are Stage* 5 *numerals for* 5, 25, 125, *and* 270 ?)

Notice that ⊙ is used *just* as a placeholder here, not also as a numeral for the number *zero*. Early numeration systems did not have a numeral for zero (that is, for the number of elements in the empty set). It was only later in history that a symbol used to hold places (as ⊙ does), also represented the number zero. Since Stage 5 is the last stage in the development of our hypothetical system, and we want this to be a numeration system for the ***whole numbers*** (the counting numbers along with zero), ⊙ will serve as a placeholder *and* as a numeral for the number zero. Thus, this dual role of ⊙ in the numeral ∧ ⊙ △ ⊙ is illustrated by writing its decimal expanded form

∧ ⊙ △ ⊙=2·125+0·25+3·5+0·1.



The *Stage* 5 expanded form of the numeral ∧ ⊙ △ ⊙ is

∧·|⊙⊙⊙+⊙·|⊙⊙+△·|⊙+⊙·|.

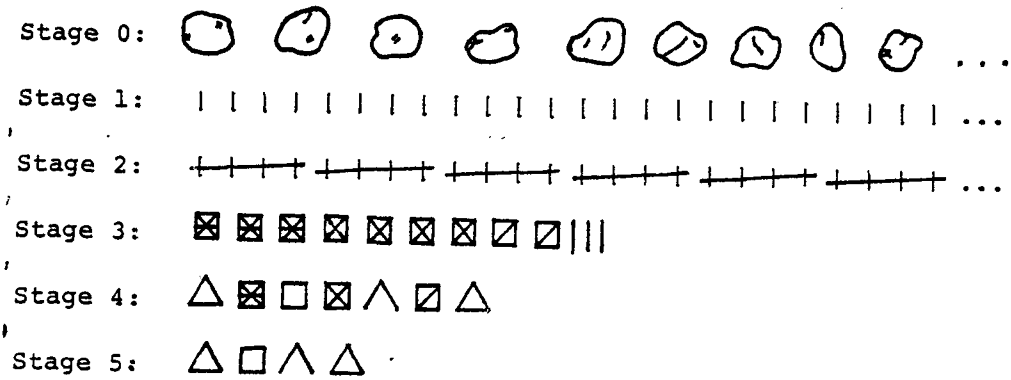
(Note that |⊙⊙⊙, |⊙⊙, and |⊙ are Stage 5 numerals for the Stage 4 numerals, ⊠, ⊠, and ⧄, respectively. *Do you see why?*) The development of our hypothetical system is concluded by stating that the digits of the system now include ⊙; thus, the set of digits is

{⊙, |, ∨, △, □}.

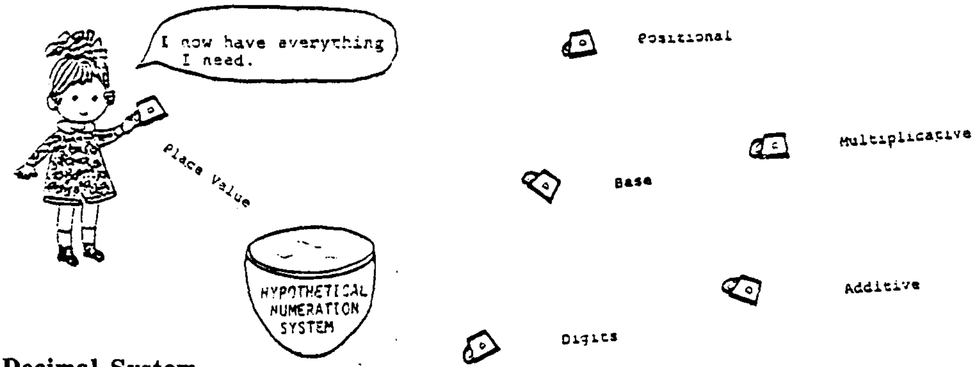
Summary of Stages 0-5

The differences among the six stages in the construction of our hypothetical numeration system are highlighted by looking at the numerals from these stages for 488. For sake of brevity, partial numerals are written for the first three stages.

This summary illustrates clearly that less writing is required as you progress from Stage 0 to Stage 5. Clearly, this is a desirable feature. However, a price is paid, since the successive numeration schemes take on more layers of abstraction. The final system evolved here is a marvelous invention, simple, yet complex. In addition to its economy in recording numerals, it eases the burden of

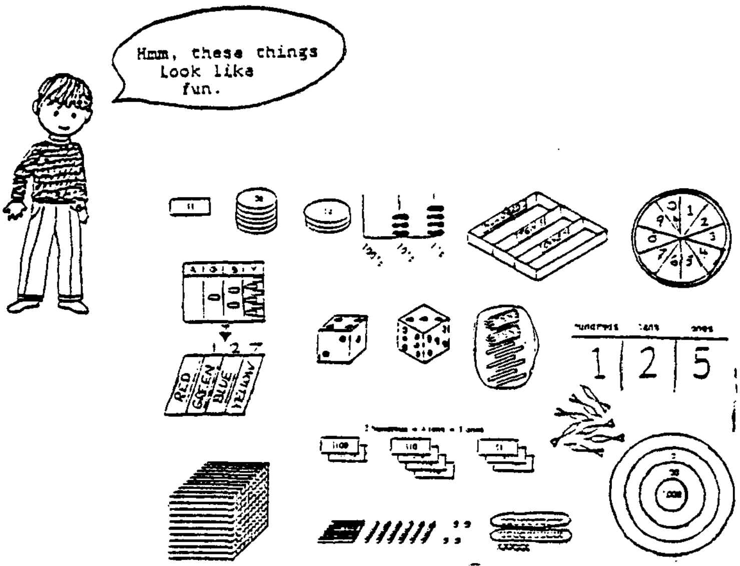


whole number computations, and has a natural extension to a numeration system for certain fractions. The ingredients of this system are summarized:



Decimal System

With a change in the base (*ten* instead of five) and a change in the symbols for the numbers



less than the base (0, 1, 2, 3, 4, 5, 6, 7, 8, 9 in place of |, ∧, △, □), ***our hypothetical system becomes the decimal system.*** Hence, as was promised, you have been presented with an evolution of the decimal system for naming whole numbers through a sense of numeration history. Hopefully, enough "sense" has been provided so that you can better plan instruction for a particular stage of decimal development (which may be closely akin to one of the Stages 0~5). Included in your plans should be a selection of appropriate manipulative aids, games, and worksheets that are available for teaching the decimal system.

Epilogue

The "sense" of numeration history that has been presented here will be especially beneficial to you if it is coupled with additional readings on the ***teaching of numeration.*** An excellent source on the teaching of the decimal system which includes many classroom activities for developing it, is the NCTM Yearbook entitled *Mathematics Learning in Early Childhood* (4). Included in this book is an important aspect of the teaching of numeration that was not touched on here, namely, the relationship between the oral names for numbers (e.g., "fifteen", "twenty-six"), and the corresponding decimal numerals.

There are many books and materials available on historical numeration systems; a few are listed in the References (1) (2) (3). It should be easier for you now to understand and place historical numeration systems, since they can be closely identified with one of the six stages presented here. Why not ***enrich*** your students by exposing them to some historical numeration systems? Some middle school worksheets on historical numeration systems are found in (3).

The decimal system of numeration—what a beautiful and important creation!

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