韓國數學敎育學會誌<數學敎育>
1982. 12. Vol. XXI. No. 1

Journal of the Korea Society of Mathematical Education
Dec. 1982. Vol. XXI. No. 1

A Common Fixed Point Theorem in Saks Spaces

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Resently, some fixed point and common fixed point theorems for commuting mappings in complete metric spaces are proved by S.S. Chang [5], K.M. Das and K.V. Naik [6], K. Iseki [7], G. Jungck [8, 9] and S.L. Singh [13, 14, 15].

In this paper,applying some common fixed point theorems of K. Iseki [7] and S.L. Singh [13] for commuting mappings in a space with two metric, we shall prove a common fixed point theorem in Saks spaces.

Some properties and definitions for Saks spaces are given in [1-4, 10-12, 16]

Let X be a linear space. A real-valued function $\|\cdot\|$ defined on X will be called a B-norm if it satisfies the following conditions:

(1) $\|x\|=0$ if and only if $x=0$,

(2) $\|x+y\|\leq\|x\|+\|y\|$,

(3) $\|\alpha x\|=|\alpha|\,\|x\|$, α being any real number.

Each real-valued function $\|\cdot\|$ satisfying the above conditions (1), (2) and the following one:

(4) if the sequence $\{\alpha_n\}$ of real numbers converges to a real number α and $\|x_n-x\|\to 0$ as $n\to\infty$, then $\|\alpha_n x_n-\alpha x\|\to 0$ as $n\to\infty$ will be said to be an F-norm.

A two-norm space is a linear space X with two norms, a B-norm $\|\cdot\|_1$ and an F-norm $\|\cdot\|_2$, and denoted by $(x, \|\cdot\|_1, \|\cdot\|_2)$.

If two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are defiined on X and $x_n\in X$, $\|x_n\|_1\to 0$ as $n\to\infty$ implies $\|x_n\|_2\to 0$, then the norm $\|\cdot\|_1$ is called non-weaker than $\|\cdot\|_2$ in X (denoted by $\|\cdot\|_2\leq\|\cdot\|_1$). If $\|\cdot\|_2\leq\|\cdot\|_1$ and $\|\cdot\|_1\leq\|\cdot\|_2$, then the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.

A sequence $\{x_n\}$ of points in a two-norm space $(x, \|\cdot\|_1, \|\cdot\|_2)$ is said to be γ-convergent to x_0 in X (denoted by $x_n\xrightarrow{\gamma}x_0$ or $\gamma\text{-}\lim_{n\to\infty} x_n=x_0$) if $\sup_n\|x_n\|_1<\infty$ and $\lim_{n\to\infty}\|x_n-x_0\|_2=0$ and a sequence $\{x_n\}$ in a two-norm space is said to be γ-Cauchy if $(x_{p_n}-x_{q_n})\xrightarrow{\gamma}0$ as $p_n, q_n\to\infty$.

A two-norm space is called γ-complete if for every γ-Cauchy sequence $\{x_n\}$ in two-norm space, there exists $x_0\in X$ such that $x_n\xrightarrow{\gamma}x_0$.

Let X be a linear set and suppose that $\|\cdot\|_1$ is a B-norm and $\|\cdot\|_2$ is an F-norm on X. Let $X_s=\{x\in X; \|x\|_1<1\}$ and define $d(x,y)=\|x-y\|_2$ for all x, y in X_s. Then d is a metric on X_s and the metric space (X_s, d) will be called a Saks set. If (X_s, d) is complete, it will be called a Saks space and denoted (X_s, d) by $(x, \|\cdot\|_1, \|\cdot\|_2)$.

In [10], W. Orlicz has proved the following:

Theorem 1. *Let $(X_s, d)=(X, \|\cdot\|_1, \|\cdot\|_2)$ be a Saks space. Then the following statements are* *equivalent:*

(1) $\|\cdot\|_1$ *is equivalent to* $\|\cdot\|_2$ *on* X.

(2) $(X, \|\cdot\|_1)$ *is a Banach space and* $\|\cdot\|_2 \geq \|\cdot\|_1$ *on* X.

(3) $(X, \|\cdot\|_2)$ *is a Frechet space and* $\|\cdot\|_1 \geq \|\cdot\|_2$ *on* X.

Now, we are ready to give our main theorem.

Theorem 2. *Let* $(X_s, d) = (X, \|\cdot\|_1, \|\cdot\|_2)$ *be a Saks space which* $\|\cdot\|_1$ *is equivalent to* $\|\cdot\|_2$ *on* X. *If two commuting mappings* S *and* T *from* X *into itself satisfy the following conditions:*

(1) $S(X) \subset T(X)$,

(2) $\|Sx-Sy\|_2 \leq \alpha\{\|Tx-Sx\|_2 + \|Ty-Sy\|_2\} + \beta\{\|Tx-Ty\|_2 + \|Ty-Sx\|_2\} + \gamma\|Tx-Ty\|_2$ *for all* x, y *in* X, *where* α, β, γ *are non-negative and* $2\alpha+2\beta+\gamma<1$,

(3) S *and* T *are continuous with respect to* $\|\cdot\|_1$.

Then S *and* T *have a unique common fixed point in* X.

Proof. From (1), we can define a sequence $\{x_n\}$ of points in X as follows: For $x_0 \in X$, let x_1 be such that $Tx_1 = Sx_0$, in general, choose x_{n+1} so that $Tx_{n+1} = Sx_n$.

Then by (2), we have $\|Tx_{n+1} - Tx_n\|_2 = \|Sx_n - Sx_{n-1}\|_2 \leq \alpha\{\|Tx_n - Sx_n\|_2 + \|Tx_{n-1} - Sx_{n-1}\|_2\} + \beta\{\|Tx_n - Sx_{n-1}\|_2 + \|Tx_{n-1} - Sx_n\|_2\} + \gamma\|Tx_n - Tx_{n-1}\|_2 \leq p\|Tx_n - Tx_{n-1}\|_2 \leq p^n\|Tx_1 - Tx_0\|$, where $0 < p = (\alpha+\beta+\gamma)/(1-\alpha-\beta) < 1$. Therefore by Theorem 1, since $\|Tx_{n+1} - Tx_n\|_2 \to 0$ as $n \to \infty$, $\|Tx_{n+1} - Tx_n\|_2 \to 0$. This shows that the sequence $\{Tx_n\}$ is a Cauchy sequence with respect to $\|\cdot\|_1$ and from Theorem 1, since $(X, \|\cdot\|_1)$ is a Banach space, $\{Tx_n\}$ has a limit point t in X, i.e., $\|Tx_n - t\|_1 \to 0$ as $n \to \infty$ and also in view of $Tx_{n+1} = Sx_n$, $\|Sx_n - t\|_1 \to 0$ as $n \to \infty$. Since S and T are continuous with respect to $\|\cdot\|_1$ we have $\|STx_n - St\|_1 \to 0$ and $\|TSx_n - Tt\| \to 0$ as $n \to \infty$. But since S and T commute, $St = Tt$, and hence,

(4) $TTt = TSt = STt = SSt$.

Therefore by (2) and (4), $St = SSt$, and so, St is a common fixed point of S and T. Uniqueness of the common fixed point of S and T follows easily.

As an immediate consequence of Theorem 2, we have the following:

Corollary 1. *Let* $(X_s, d) = (X, \|\cdot\|_1, \|\cdot\|_2)$ *be a Saks space which* $\|\cdot\|_1$ *is equivalent to* $\|\cdot\|_2$ *on* X. *If two commuting mappings* S *and* T *from* X *into itself satisfy the following conditions:*

(1) $S^n(X) \subset T(X)$ *for a positive integer* n,

(2) $\|S^n x - S^n y\|_2 \leq \alpha\{\|Tx - S^n x\|_2 + \|Ty - S^n y\|_2\} + \beta\{\|Tx - S^n y\|_2 + \|Ty - S^n x\|_2\} + \gamma\|Tx - Ty\|_2$ *for all* x, y *in* X, *where* α, β, γ *are non-negative and* $2\alpha+2\beta+\gamma<1$,

(3) S *and* T *are continuous with respect to* $\|\cdot\|_1$.

Then S *and* T *have a unique common fixed point in* X.

In case T is an identity mapping, the following corollary is obtained as a special case of our theorem.

Corollary 2. *Let* $(X_s, d) = (X, \|\cdot\|_1, \|\cdot\|_2)$ *be a Saks space which* $\|\cdot\|_1$ *is equivalent to* $\|\cdot\|_2$. *If a mapping* S *from* X *into itself satisfies the following conditions:*

(1) $\|Sx-Sy\|_2 \leq \alpha\{\|x-Sx\|_2 + \|y-Sy\|_2\} + \beta\{\|x-Sy\|_2 + \|y-Sx\|_2\} + \gamma\|x-y\|_2$, *for all* x, y *in* X, *where* α, β, γ *are non-negative and* $2\alpha+2\beta+\gamma<1$,

(2) *S is continuous with respect to* $\|\cdot\|_1$.

Then S has a unique fixed point in X.

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