韓國數學教育學會誌〈數學教育〉
1982. 12. Vol. XXI. No. 1

Journal of the Korea Society of Mathematical Education
Dec. 1982. Vol. XXI. No. 1

Single Generators For Von Neumann Algebras And C*-Algebras

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I. Preliminary.

C. Pearcy shows that type I von Neumann algebras acting on a separable Hilbert space are singly generated, and W. Wogen shows that properly infinite von Neumann algebras acting on a separable Hilbert space are singly generated. But for the case of type II_1 von Neumann algebras, only partial answers were given for the single generator problem by C. Pearcy. We will consider the various forms of single generators of properly infinite von Neumann algebras acting on a separable Hilbert space and consider some C*-algebras which are singly generated.

In this paper we assume that H is a separable Hilbert space and by an operator we mean a bounded linear operator on a Hilbert space. A von Neumann algebra $\mathscr{A}$ acting on a Hilbert space is said to be generated by a family $\{A, B, \cdots\}$ of operators if it is the smallest von Neumann algebra containing each member of the family $\{A, B, \cdots\}$ and it is denoted by $\mathscr{A}=R(A, B, \cdots)$. And we use $\mathscr{A}'$ for the commutant of a von Neumann algebra $\mathscr{A}$. For a von Neumann algebra $\mathscr{A}$, we denote by $M_n(\mathscr{A})=\mathscr{A}\otimes M_n$ the algebra of all $n\times n$ matrices with entries from $\mathscr{A}$ and if $C\in M_n(\mathscr{A})$ is a diagonal matrix with diagonal entries $C_1, C_2, \cdots, C_n\in\mathscr{A}$ we denote this by $C=\text{diag}(C_1, C_2, \cdots, C_n)$ and if $C_1=C_2=\cdots=C_n=B$, we denote this by $C=\text{diag}_n(B)$.

II. Single generators for von Neumann algebras.

Lemma 1. *Let $\mathscr{A}$ be a properly infinite von Neumann algebra. Then $\mathscr{A}$ is spatially *-isomorphic to $M_n(\mathscr{A})$ for all positive integer n.*

Proof: See Topping.

Lemma 2. *Let $\mathscr{A}$ be a von Neumann algebra and let $T\in M_n(\mathscr{A})$. If $C^*=C\in R(T)'$ is of the form $diag_n(D)$ with $D\in\mathscr{A}'$, then $R(T)=M_n(\mathscr{A})$.*

Proof. Let $A=(A_{i,j})^n_{i,j=1}\in M_n(\mathscr{A})$.
To show $AC=CA$ for all $C\in R(T)'$, it suffice to show that $AC=CA$ for all $C^*=C\in R(T)'$. Let $C=\text{diag}_n(D)$ with $D\in R(T)'$. Then

$$AC=(A_{i,j}D)^n_{i,j=1}=(DA_{i,j})^n_{i,j=1}=CA.$$

Hence we have the required conclusion.

Theorem 1. *Any properly infinite von Neumann algebra $\mathscr{A}$ acting on a separable Hilbert space H is generated by a partial isometry T such that TT^* and T^*T commute.*

Proof. We show that $M_3(\mathscr{A})$ is generated by a partial isometry $T\in M_3(\mathscr{A})$ such that TT^* and T^*T commute. Since $\mathscr{A}$ is properly infinite we can choose a single generator $A\in\mathscr{A}$ such that A is invertible and $1-A^*A$ is positive invertible. Now let $B=(1-A^*A)^{1/2}$, $C=(A^*A)^{1/2}$, $D=-ABC^{-1}$, and consider

$$T=\begin{pmatrix} 0 & B & C \\ 0 & A & D \\ 0 & 0 & 0 \end{pmatrix}$$

Then $TT^*=\mathrm{diag}(1,1,0)$, $T^*T=\mathrm{diag}(0,1,1)$, so T is a partial isometry such that TT^* and T^*T commute. Since any $E^*=E\in R(T)'$ is of the form $\mathrm{diag}_3(U)$ with $U\in\mathscr{A}'$, by Lemma 2 we conclude that $R(T)=M_3(\mathscr{A})$.

Definition. A partial isometry T will be called of *class n* if $T, T^2, \cdots, T^n$ are partial isometries.

Theorem 2. *For any given integer $n>0$, any properly infinite von Neumann algebra $\mathscr{A}$ acting on a separable Hilbert space is generated by a partial isometry T of class n.*

Proof. We use induction on n. Choose a proper contractive invertible operator $A\in\mathscr{A}$ which generates A and let $B=(1-A^*A)^{1/2}$. Consider

$$T=\begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix}$$

Then $T^*T=\mathrm{diag}(1,0)$ and $1-T^*T$ is equivalent to 1. And $C^*=C\in R(T)'$ is $\mathrm{diag}_2(D)$ for some $D\in\mathscr{A}'$. So that $R(T)=M_2(\mathscr{A})$ and T is a partial isometry. Now assume that A is a partial isometry of class n which generates $\mathscr{A}$ such that $1-A^*A$ is equivalent to 1. Now consider $T\in M_2(\mathscr{A})$ as above with $BB^*=1$, $B^*B=1-A^*A$. Then $T^*T=\mathrm{diag}(1,0)$ and any $C^*=C\in R(T)'$ is $\mathrm{diag}_2(D)$ for some $D\in\mathscr{A}'$ and for $k=1,2,\cdots,n+1$, $(T^k)^*T^k=\mathrm{diag}\ ((A^{k-1})^*A^{k-1},0)$. So that $M_2(\mathscr{A})$ is generated by a partial isometry of class n.

Lemma 3. *Let K be a compact subset of the unit circle with nonempty interior. If type II_1 summand of a von Neumann algebra $\mathscr{A}$ acting on a separable Hilbert space is singly generated, then the von Neumann algebra $\mathscr{A}$ is singly generated by an operator A which is similar to a unitary operator and $\|A\|<1+\varepsilon$, $\sigma(\mathscr{A})\subset K$.*

Proof. See Wogen(6).

Theorem 3. *Let K be a compact set in plane with interior point $z_0(\neq 0)$. Then for given $\varepsilon>0$, any von Neumann algebra $\mathscr{A}$, which has singly generated type II_1 summand, has a single generator which is similar to normal and $\sigma(A)\subset K$ $\|A\|<|z_0|+\varepsilon$.*

Proof. We may assume that K is a closed disk centered at z_0. Let $K_1=\frac{1}{\lambda}K\cap U$, where $\lambda=|z_0|$, U is the unit cicle. Then we can apply the Lemma 3, to the compact set K_1 and ε/λ. So we have a single generator A_1 for $\mathscr{A}$ which is similar to a unitary operator U_1 and $\sigma(A_1)\subset K_1$ $\|A_1\|\leqq 1+\varepsilon/\lambda$. Let $A=\lambda A_1$ then $R(A)=\mathscr{A}$. A is similar to a normal operator λU_1 and $\sigma(A)\subset K$. $\|A\|<\lambda+\varepsilon$.

III. Single Generators for C^*-Algebras.

A C^*-algebra $\mathscr{A}$ is generated by a family $\{A,B,C,\cdots\}\subset\mathscr{A}$ as a C^*-algebra if it is the smallest C^*-algebra containg $\{A,B,C,\cdots\}$, and we denote $\mathscr{A}=C^*(A,B,C,\cdots)$. We denote $\mathscr{K}$ by a C^*-algebra of compact operators on a separable Hilbert space H which is infinite dimensional. We use the fact that any separable C^*-algebra is generated as a C^*-algebra by a countable set of self-adjoint elements in the C^*-algebra.

Theorem 3. *If $\mathscr{A}$ is a separable C^*-algebra then the C^*-tensor product $\mathscr{A}\otimes\mathscr{K}$ is singly generated as a C^*-algebra.*

Proof. Since $\mathscr{A}$ is separable, we can find a countable set $\{A_n|n=1,2,3,\cdots\}$ of self adjoint elements of $\mathscr{A}$ which generates $\mathscr{A}$ as a C^*-algebra. By multiplying by scalar and translating by scalar multiples of $1_{\mathscr{A}}$, we can assume that $\sigma(A_n)\subset[2^{-2n-1}, 2^{-2n}]$ for each n. Then each A_n is positive and invertible.

First we show that $\mathscr{A}\otimes\mathscr{K}$ is generated by two elements $A, B\in \mathscr{A}\otimes\mathscr{K}$ as a C^*-algebra. Let $\{e_n\}$ be an orthonormal basis for H. Let $E_{i,j}\in \mathscr{K}$ be defined by $E_{i,j}(e_j)=e_i$ and $E_{i,j}(e_k)=0$ if $k\neq j$. Then

$$\|A_n\times E_{n,n}\|=\|A_n\| \ \|E_{n,n}\|\leqq 2^{-2n}.$$

So that the series $\Sigma_{n=1}^{\infty}A_n\otimes E_{n,n}$ converges in norm to a positive element $A\in\mathscr{A}\otimes\mathscr{K}$. Let $S\in\mathscr{K}$ be the weighted backward shift defined by

$$S_{e_n}=(n-1)^{-1}e^{n-1} \text{ for } n\geq 2 \text{ and } S_{e_1}=0,$$

and let $B=1_{\mathscr{A}}\otimes S$. Since S is irreducible, $C^*(S)=\mathscr{K}$. Thus $1_{\mathscr{A}}\otimes K\in C^*(A,B)$ for each $K\in\mathscr{K}$. Since

$$A_m\otimes E_{1,1}=(1_{\mathscr{A}}\otimes E_{1,m})(1_{\mathscr{A}}\otimes E_{m,n})A(1_{\mathscr{A}}\otimes E_{m,1})$$

we conclude that $A_m\otimes E_{1,1}\in C^*(A,B)$ for each m. Therefore $F\otimes E_{1,1}\in C^*(A,B)$ for each $F\in\mathscr{A}$. But then for any $F\in\mathscr{A}$

$$F\otimes E_{i,j}=(1_{\mathscr{A}}\otimes E_{i,1})(F\otimes E_{1,1})(1_{\mathscr{A}}\times E_{1,j})$$

is in $C^*(A,B)$, and linear combinations of such elements are dense in $\mathscr{A}\otimes\mathscr{K}$, we have $\mathscr{A}\otimes\mathscr{K}=C^*(A,B)$.

Now we note the following isometric *-isomorphisms of C^*-tensor products

$$(\mathscr{A}\otimes\mathscr{K})\otimes M_2\simeq\mathscr{A}\otimes(\mathscr{K}\otimes M_2)\simeq\mathscr{A}\otimes\mathscr{K}.$$

Thus it suffice to show that $(\mathscr{A}\otimes\mathscr{K})\otimes M_2$ is singly generated as a C^*-algebra. Consider

$$T=\begin{pmatrix} B & A \\ 0 & 0 \end{pmatrix}$$

Then

$$TT^*=\begin{pmatrix} AA^*+BB^* & 0 \\ 0 & 0 \end{pmatrix}$$

Now

$$AA^*+BB^*=\Sigma(A_n{}^2\otimes E_{n,n})+1_{\mathscr{A}}\otimes(\Sigma n^{-2}E_{n,n})$$
$$=\Sigma(A_n{}^2+n^{-2}1_{\mathscr{A}})\otimes E_{n,n}$$

Note that $\sigma(A_n{}^2+n^{-2}1_{\mathscr{A}})\subset[2^{-4n-2}+n^{-2}, 2^{-4n}+n^{-2}]$ and these are disjoint intervals.

And so

$$\sigma(AA^*+BB^*)=\sum_{n=1}^{\infty}\sigma(A_n{}^2+n^{-2}1_{\mathscr{A}})\cup\{0\}.$$

Thus the characteristic function of $\sigma(A_n{}^2+n^{-2}1_{\mathscr{A}})$ is continuous on $\sigma(AA^*+BB^*)$. So by the functional calculus, we obtain $1_{\mathscr{A}}\otimes E_{n,n}\in C^*(AA^*+BB^*)$ for each n. Let $U_k=1_{\mathscr{A}}\otimes\Sigma_{n=1}^{k}E_{n,n}\in C^*(AA^*+BB^*)$. Then $\{U_k\}$ forms a positive approximate identity for $\mathscr{A}\otimes\mathscr{K}$. So we have

$$\lim\begin{pmatrix} B & A \\ 0 & 0 \end{pmatrix}\begin{pmatrix} U_k & 0 \\ 0 & 0 \end{pmatrix}=\begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}\in C^*(T).$$

Thus

$$\begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} AA' & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \text{ are in } C^*(T).$$

Hence if $V_k=\Sigma_{n=1}^{k}A_n{}^{-1}\otimes E_{n,n}$, then for each k

$$\begin{pmatrix} V_k & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & U_k \\ 0 & 0 \end{pmatrix} \in C^*(T).$$

So that $(\mathcal{A} \otimes \mathcal{K}) \otimes M_2$ is generated by the set

$$\left\{ \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & U_k \\ 0 & 0 \end{pmatrix}, : k=1,2,\cdots \right\}$$

And we have the required conclusion.

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