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A property of Dedekind domain

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1. Introduction

Historically, Dedekind domains are arose in number theory as the ring of all integral elements in some finite algebraic extention of the rational numbers. We know that an integral domain R is called a *Dedekind domain* if every proper ideal of R is a product of prime ideals. And Dedekind domains are characterized in several different ways.

In this paper, we shall see that an integral domain R is a Dedekind domain if and only if every ideal of R may be generated by two elements.

2. Preliminaries

(1) A PID is a Dedekind domain.

(2) Let R be an integral domain. Then R is a Dedekind domain if and only if R is noetherian and R_P is a PID for every prime ideal P of R.
Moreover, R_P is a Dedekind semilocal domain, and so R_P is PID.

(Lemma) *If R is a Dedekind domain and I is a proper integral ideal of R, then every ideal of R/I is principal.*

Proof: Let $I=\prod P_i$ and let $S=R-\bigcup P_i$. Then $R/I\approx R_s/IR_s$ and R_s is PID. It follows that every ideal of R_s/IR_s is principal, and so the same is true of R/I.

3. Main Theorem

Theorem: *If R is an integral domain, then R is a Dedekind domain if and only if given any nonzero element x in an ideal I of R, there is a y in I such that $(x,y)=I$.*

Proof: We take a non-zero element x in I. By Lemma, R/Rx is a PID, and so I/Rx is principal. Let y be an element of I whose residue class modulo Rx generates I/Rx. Therefore, $(x,y)=I$. Conversely, it sufficies to show that R_P is a PID for each prime ideal P of R. Let P be a nonzero prime ideal of R and let $M=PR_P$ be the maximal ideal of R_P. Then $0\neq M^2\subset M$. Since P is generated by two elements in R, M is generated by two elements in R_P. Take an x in M^2. By hypothesis, there is a y in M such that $(x,y)=M$. This implies M is actually principal which in turn implies R_P is PID.

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