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# Best Approximation In Non-Archimedean Quasi Normed Spaces

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In this note we have discussed the problem of best approximation in non-archimedean (n.a.) quasi normed spaces. It has been observed that most of the results in n.a. quasi normed spaces should come automatically from those in n.a. normed spaces.

To start with we recall the notion of n.a. quasi-normed space, introduced by K.Iseki [3].

**Definition 1.** Let $X$ be a linear space over a valued field $K$. A mapping $\|\cdot\|: X \to R$ is called a n.a. quasi-norm with power $r$ if it satisfies the following axioms:

(i) $\|x\|=0$ if and only if $x=0$

(ii) $\|x+y\| \leq \max\{\|x\|, \|y\|\}$ for all $x, y \in X$

(iii) $\|\alpha x\| = |\alpha|^r \|x\|$ for $\alpha \in K$ and $x \in X$ ($r$ real, $0<r<\infty$)

It is easy to see that if quasi-norm of $X$ is n.a., then the valuation of $K$ is also n.a.

T. Konda [4] has given some results in n.a. quasi-normed spaces similar to those for n.a. normed spaces. A natural temptation is to develope a theory of n.a. quasi-normed spaces parallel to that of n.a. normed spaces. It appears from the following theorem that the structure of n.a. quasi-normed spaces is quite similar to that of n.a. normed spaces and so many results in these spaces should come automatically from those in n.a. normed spaces.

**Theorem 1.** *If* $(X, \|\cdot\|)$ *is a n.a. normed space, then* $X$ *together with the function* $p$ *defined by* $p(x)=\|x\|^r$ *is a n.a. quasi-normed space with power* $r$. *Conversely, if* $(X, p)$ *is a n.a. quasi-normed space with power* $r$, *then* $X$ *together with* $\|\cdot\|$ *defined by* $\|x\|=[p(x)]^{1/r}$ *is a n.a. normed space.*

**Proof.** It is evident that we need verify only the third property.

Suppose $(X, \|\cdot\|)$ is a n.a. normed space. Let $\alpha \in K$ and $x \in X$. Consider

$$p(\alpha x)=\|\alpha x\|^r=(|\alpha| \ \|x\|)^r=|\alpha|^r \ \|x\|^r=|\alpha|^r p(x).$$

So $(X, p)$ is a n.a. quasi-normed space with power $r$.

Now suppose $(X, p)$ is a n.a. quasi-normed space with power $r$. Then

$$\|\alpha x\|=[p(\alpha x)]^{1/r}=[|\alpha|^r p(x)]^{1/r}=|\alpha|[p(x)]^{1/r}=|\alpha| \ \|x\|$$

So $(X, \|\cdot\|)$ is a n.a. normed space.

It was shown in [6] that the following problem of best approximation in n.a. quasi-normed spaces has a solution.

Let $X$ be a n.a. quasi-normed space over a n.a valued field $K$ and let $y$ be an arbitrary element in $X$. Given a finite system of linearly independent elements $x_1, x_2, \ldots, x_m$. Does there exist $\alpha_i$'s in

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$K$ such that $\|y-\alpha_1 x_1-\alpha_2 x_2-\cdots-\alpha_m x_m\|$ is minimum?

Now discuss the general problem of best approximation in n.a. quasi-normed spaces.

**Definition** 2. Let $X$ be a n.a. quasi-normed space over a n.a. valued field $K$. Let $E\subset X$ be a closed linear subspace of $X$. For a given $x\epsilon X$, a best approximation of $x$ in $E$ is defined to be an element $\xi\epsilon E$ such that

$$\|x-\xi\|=\inf_{g\in E}\|x-g\|.$$

This definition is same as that in archimedean analysis. The main problems to be discussed regarding best approximation in n.a. quasi-normed spaces are:

(1) The existence of best approximation for an arbitrary $x\epsilon X$ and for all $E\subset X$ or for a given $E\subset X$.

(2) The problem of uniqueness of best approximation.

Answer to the second problem is given by the following theorem:

**Theorem** 2. *A best approximation of $x\epsilon X$, $x\notin E$ in $E$ when it exists is never uniquely determined unless $E=\{0\}$.*

The proof is similar to that given in [5] for n.a. mormed spaces.

**Remark.** We know that in spaces over $\boldsymbol{R}$ or $\boldsymbol{C}$, a necessary and sufficient condition of uniqueness of best approximation is that the norm be strict, *i.e.*, $\|x+y\|=\|x\|+\|y\|$ and $x\neq 0$ imply $y=tx$ for some $t\geq 0$ (cf. [1]). A definition of a strict norm tor n.a. quasi-normed spaces similar to that in spaces over $\boldsymbol{R}$ or $\boldsymbol{C}$ has no sense, for the equality $\|x+y\|=\|x\|+\|y\|$ leads us to the result that $x$ or $y$ must be zero. In this way no quasi-norm on n.a. quasi-normed space is strict. One could say that this is in agreement with Theorem 2.

Regarding the existence of best approximation we have the following result.

**Theorem** 3. *Let $Y$ be a subspace of a n.a. quasi-normed space $X$. If $Y$ is spherically complete, then for any $x\epsilon X$. there is some $y\epsilon Y$ such that $d(x,y)=d(x,Y)=\inf_{y\in Y} d(x,y)$.*

In fact the result holds in the more general case where $Y$ is a spherically complete subspace of an ultrametric space $X$ (cf. [7]).

Next we discuss the following problem of best approximation in n.a. quasi-normed spaces. This problem for n.a. normed space was raised by Monna in [5] and its partial answer was given by Ikada and Haifawi in [2].

**Problem.** *Let $E$ be a closed subspace of a n.a. quasi-normed space $X$. Let $x_0\epsilon X$, $x_0\notin E$. Suppose best approximation of $x_0$ in $E$ exists.* (*)

*(Under what conditions) Does every $x\epsilon X$ has a best approximation in $E$?*

If we replace (*) by the following condition:

For any closed proper subspace $E$ of $X$, there exists at least one element $x\epsilon X$, $x\notin E$ which has a best approximation in $E$. (**)

We shall see that under the condition (**) for any closed subspace $E$ of $X$, every element in $X$ has a best approximation in $E$. We shall also give an example to show that the condition (*) is not sufficient for the existence of best approximation in $E$ for every $x\epsilon X$.

The answer to the first assertion is given by Theorem 4 which is based on the following two Lemmas.

**Lemma 1.** *Let $E$ be a closed subspace of a n.a. quasi-normed space $X$. If $x_0 \in X$, $x_0 \notin E$ has a best approximation in $E$, then every element in $\{x_0, E\}$ has a best approximation in $E$.*

**Lemma 2.** *Let $E, G$ be two subspaces of $X$, a n.a. quasinormed space, $E \subset G$. If $x \in X$ has a best approximation in $G$ and if every element in $G$ has a best approximation in $E$, then $x$ has a best approximation in $E$.*

**Theorem 4.** *Let $X$ be a n.a. quasi-normed space over a field $K$ satisfying the condition (**). Then for any closed subspace $E \subset X$ every element in $X$ has a best approximation in $E$.*

The proofs of Lemmas 1 and 2 and Theorem 4 are similar to those given in [2] for n.a. normed spaces.

The following example given in [2] for n.a. normed spaces, extended trivially to n.a. quasi-normed spaces, shows that condition (*) is not sufficient for the existence of best approximation in $E$ for every $x \in X$.

**Example.** Let $X$ be the space of all formal power series such that every non zero element $x \in X$ is of the form $x = a_1 t^{\alpha_1} + a_2 t^{\alpha_2} + \cdots\cdots$, where $\alpha_i$'s are rational numbers well ordered in natural (ascending) order and the $a_i$'s are non zero co-efficients taken from some field $F$. $X$ is a n.a. valued field with $|x| = e^{-\alpha_1}$, $(a_1 \neq 0)$ and may be regarded as a n.a. quasi-normed space of power $r$ over a subfield $K$ (the quasi norm being defined as $\|x\| = e^{-\alpha_1 r}$), with elements such that $\{\alpha_i\}$ is a finite set or sequence tending to infinity. $K$ is closed subfield of $X, K$ is complete but not spherically complete. Let $u_0$ be an element of $X, u_0 \in K$, so that $u_0 = a_1 t^{\alpha_1} + a_2 t^{\alpha_2} + \cdots\cdots$, where $\alpha_1 < \alpha_2 < \cdots\cdots$ is a sequence which converges to an irrational number. Then for $x \in K$, $\|u_0 - x\| = e^{-\alpha_{n+1}^{r}}$ for some $n$ and $\inf_{x \in K} \|u_0 - x\| = e^{-r}$ But there is no element $\xi \notin K$ so that $\|u_0 - \xi\| = e^{-r}$. Hence $u_0$ has no best approximation in $K$.

Let $u_1$ be an element of $X/K$ such that $\|u_1\| = \alpha \in \|K\|$. Consider the vector space $W = K + Ku_0 + Ku_1$. Let us define a quasi norm $\|\cdot\|^*$ on $W$ as follows.

For $y \in W$, $y = x + \lambda u_1$ where $x \in K + Ku_0$, $\lambda \in K$, $\|y\|^* = \|x + \lambda u_1\| = \max\ \{\|x\|, \|\lambda\| \alpha\}$. Then $(W, \|\cdot\|^*)$ is a n.a. quasi normed space of power $r$.

Now we show that $u_1$ has a best approximation in $K$. For this consider $\inf_{y \in K} \|u_1 - y\|^* = \inf_{y \in K} \max \{\|u_1\|, \|y\|\} = \alpha$. Let $y_0 \in K$ be such that $\|y_0\| < \alpha$. Then $\|u_1 - y\|^* = \alpha$. Therefore $u_1$ has a best approximation in $K$ but not every element of $W$ has a best appriximation in $K$.

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