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A Note on Choquet boundaries for function algebras

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1. Introduction

Let X and Y be compact Hausdorff spaces. Throughout this note, $C(X)$ and $C(Y)$ will denote the algebras of all complex-valued continuous functions on X and Y respectively.

R. Phelps [3] proved various properties of the Choquet boundary ∂M for a linear subspace M of $C(X)$ with $1 \in M$. The relation between the extreme points of $K_2(A,B)$ and the multiplicative linear transformations in $K_2(A,B)$ was discussed in [4], where A and B are subalgebras of $C(X)$ and $C(Y)$ respectively, both of which contain the constant functions.

Note that the state space $K(A)$ for A coincides with $K_2(A,B)$ when B is the algebra of complex numbers. The purpose of this note is to clarify how ∂A is related to ∂B when A is algebraically homomorphic to B.

2. Basic concepts

Let A and B be subalgebras of $C(X)$ and $C(Y)$ respectively, and assume that both A and B contain the constant functions. Let $L(A,B)$ be the set of all linear transformations from A into B, and let $K_1(A,B)=\{T: T\in L(A,B) \text{ and } T\geq 0 \text{ and } T1=1\}$, where $T\geq 0$ means $Tf\geq 0$ whenever $f\in A$ and $f\geq 0$, and 1 represents the function which equals 1 at each point. The functions in A and B are bounded and hence these algebras have the supremum norms. Let $BL(A,B)$ be the set of T in $L(A,B)$ such that

$$\|T\|=sup\{\|Tf\|: \|f\|\leq 1,\ f\in A\}<\infty,$$

and let

$$K_2(A,B)=\{T: T\in BL(A,B), \|T\|=1=T1\}.$$

A subalgebra A of $C(X)$ is said to be ***self-adjoint***, if $\bar{f}\in A$ whenever $f\in A$ (where $\bar{f}(x)=\overline{f(x)}$ for x in X). It is easily seen that both of the above sets K are convex (that is, $\lambda T_1+(1-\lambda)T_2\in K$ whenever $T_1, T_2\in K$ and $0\leq\lambda\leq 1$). An element T of the convex set K is said to be an ***extreme point*** of K, provided that $T=\frac{1}{2}(T_1+T_2)$ for $T_1, T_2\in K$ implies that $T=T_1=T_2$. Equivalently, T is an extreme point of K iff $T\pm U\in K$ for some U in $L(A,B)$ implies $U=0$. An element T of $L(A,B)$ is ***multiplicative***, if $Tfg=TfTg$ whenever $f,g\in A$.

If B is the algebra of scalars, then $K_i(A,B)$, $i=1,2$, become sets of linear functionals, which we abbreviate by $K_i(A)$. The analogous set of linear functionals on B is denoted by $K_i(B)$.

We denote the ***evaluation functional*** at x in X by φ_x, defined by $\varphi_x f=f(x)$ for each f in A.

Note that φ_x is multiplicative in $K_i(A)$, $i=1,2$. Denoting by A_R the set of real functions in
we see that A is self-adjoint iff $A=A_R+iA_R$.

Proof of the following lemma appears in [4].

Lemma 1. *Always, $K_2(A,B)\subset K_1(A,B)$. These sets are equal iff A is self-adjoint.*

Proof of the next lemma is a slight variation of Tate's result for the real case [6].

Lemma 2. *Let i denote 1 or 2. If A is self-adjoint, then every multiplicative element of $K_i($ is an extreme point of $K_i(A)$.*

Proof. Suppose that L is a multiplicative element of $K_i(A)$ and that $L=\frac{1}{2}(L_1+L_2)$, L_1, L_2 $K_i(A)$. It suffices to show that $L=L_1=L_2$ on A_R. Since each $L_j\geq 0$, $j=1,2$, $(L_jf)^2\leq L_j(f^2)$ f each f in A_R. For, if $f\in A_R$, then $\|f\|\cdot 1-f\geq 0$, so $\|f\|-L_jf=L_j(\|f\|\cdot 1-f)\geq 0$, which shows th L_jf is real. By Tate's argument [6] (he considers the discriminant of the quadratic $0\leq L_j((\lambda f$ $1)^2)=\lambda^2L_j(f^2)+2\lambda L_jf+1$ in λ, $L_j\geq 0$ implies that $(L_jf)^2\leq L_j(f^2)$ for each f in A_R.

Hence, for such f, we have $\frac{1}{4}(L_1f)^2+\frac{1}{2}L_1f+\frac{1}{4}(L_2f)^2=(Lf)^2=L(f^2)=\frac{1}{2}L_1(f^2)+\frac{1}{2}L_2(f^2)$ $\frac{1}{2}(L_1f)^2+\frac{1}{2}(L_2f)^2$, so that $(L_1f-L_2f)^2\leq 0$. This shows that $L_1f=L_2f$ and completes the proof

Now, we prove the next theorem which plays an important role in proving Theorem 6 and 7.

Theorem 3. *Let X and Y be compact Hausdorff spaces, and let A be self-adjoint. Then $K_1(A,$ $=K_2(A,B)$ and the following assertions about an element T of $L(A,B)$ are equivalent:*

(i) *T is multiplicative and is in $K_1(A,B)$.*

(ii) *There exists a continuous functions $\xi: Y\to X$ such that $Tf=f\circ\xi$ for all f in A.*

Proof. The fact that $K_1(A,B)=K_2(A,B)$ comes from Lemma 1 and the fact that A is se adjoint. It is obvious that (ii) implies (i) from the fact that $K_1(A,B)=K_2(A,B)$.

To prove the converse, suppose that T is multiplicative, that y in Y, and consider $\varphi_y\circ T$. Th is multiplicative and is in $K_1(A)=K_2(A)$, so by Lemma 2 it is an extreme point of $K_2(A)$. the Arens-Kelley theorem [1] (see [2, p. 278] for the complex case), there exists a unique poi in X, which we denote by $\xi(y)$, such that $\varphi_y\circ T=\varphi_{\xi(y)}$. This defines ξ for each y in Y; to see th ξ is continuous, one uses the fact that the topology of X is the same as the "weak" topolo induced on it by $C(X)$, and the fact that Tf is continuous on Y for each f in A.

3. Choquet boundaries for function algebras

Suppose that M is a linear subspace (not necessarily closed) of $C(X)$ and that $1\in M$. Denote $K(M)$ (called the *state space* of A(cf. [5])) the set of all L in M^* such that $L(1)=1=\|L\|$, whe M^* is the dual of M. If we consider M^* in its weak* topology, then $K(M)$ is a nonempty comp convex subset of the locally convex space $C(X)^*$.

Note that if $L(1)=1=\|L\|$ for $L\in C(X)^*$, then $L\geq 0$ (that is, $Lf\geq 0$ whenever $f\geq 0$) [3].

Consider the evaluation functional φ_x of x in X, and let φ_x be the element of $K(M)$ defined $\varphi_xf=f(x)$, f in M. Note that φ is one-to-one, and hence is a homeomorphism, embedding X a compact subset of $K(M)$.

The intersection of all convex sets containing a subset E of $C(X)^*$ is a convex set which contains E and which is contained in every convex set containing E. This set is called the ***convex hull*** of E. The intersection of all closed (with respect to the weak* topology) convex sets containing E is a closed convex set which contains E and which is contained in every closed convex set containing E. This set is called the ***closed convex hull*** of E.

Lemma 4. *Suppose that M is a subspace of $C(X)$ and $1 \in M$. Then $K(M)$ equals the weak* closed convex hull of the set φX of all evaluation functionals φ_x at x in X.*

Proof. R. Phelps [3].

Definition Suppose that M is a linear subspace of $C(X)$ and $1 \in M$. Let ∂M be the set of all x in X for which φ_x is an extreme point of $K(M)$. We call ∂M the ***Choquet boundary*** for M.

The following lemma shows that the Choquet boundary for M makes sense.

Lemma 5. *An element L in $K(M)$ is an extreme point of $K(M)$ iff $L=\varphi_x$ for some x in ∂M.*

Proof. The "if" part of this assertion comes from the definition of ∂M. To prove the converse, suppose that L in $K(M)$ is an extreme point of ∂M. By Lemma 4, L is an extreme point of the weak* closed convex hull of φX. The fact that L is contained in φX comes from Milman's converse [3] and the fact that φX is a compact subset of $K(M)$. Then there exists x in X such that $\varphi_x = L$ and L is an extreme point of $K(M)$.

It is known that if M is all of $C(X)$, then φX is the set of all extreme points of $K(M)$, equivalently, $\partial M = X$.

Definition By a *function algebra* in $C(X)$, we mean any closed subalgebra of $C(X)$ which contains the constant functions and separates points of X.

4. Theorems

Throughout this section, let A and B be self-adjoint function algebras in $C(X)$ and $C(Y)$ for compact Hausdorff spaces X and Y respectively. In fact, $K_2(M)$ in section 2 coincides with $K(M)$ in section 3, when M is a function algebra in $C(X)$ and $C(Y)$. Thus we can express these two by $K(M)$ without confusion.

Theorem 6. *A linear transformation T in $K_2(A,B)$ is multiplicative iff there exists a continuous function $\xi: Y \to X$ such that*

(i) $Tf = f \circ \xi$ for all f in A, and

(ii) for each y in ∂B, there exists a point x in X with $\xi(y)=x$ such that $\varphi_y \circ T = \varphi_x$.

Proof. Suppose that T in $K_2(A,B)$ is multiplicative. Let ξ be the function given by Theorem 3. We are only to show that ξ satisfies the condition (ii). Suppose that y is in the Choquet boundary ∂B for B, that is, the evlatuation functional φ_y is an extreme point of $K(B)$. By the condition (i), $(\varphi_y \circ T)(f) = (Tf)(y) = (f \circ \xi)(y) = \varphi_{\xi(y)} f$ for all f in A. Choose $\xi(y)=x$ for some x in X and then $\varphi_y \circ T = \varphi_x$.

Now we will show that the point x is in ∂A. By Lemma 5, it is enough to show that φ_x is an extreme point of $K(A)$. It is obvious that φ_x is in $K(A)$. Since T is multiplicative in $K_2(A,B)$,

$\varphi_y \circ T = \varphi_x$ is also multiplicative in $K(A)$. It follows from Lemma 2 that φ_x is an extreme point of $K(A)$.

The converse is obvious from Theorem 3.

The next theorem is similar to the above as the reversed form.

Theorem 7. *Suppose that T in $K_2(A,B)$ is multiplicative and that ξ given by Theorem 3 is surjective. Then for each x in ∂A, there exists a point y in ∂B with $\xi(y)=x$ such that $\varphi_y \circ T = \varphi_x$.*

Proof. Suppose that x is in the Choquet boundary ∂A for A. Since ξ is surjective, we can choose a point y in Y such that $\xi(y)=x$. Hence we can choose the evaluation functional φ_y defined by $\varphi_y g = g(y)$ for all y in B. This implies $\varphi_y \circ T = \varphi_x$.

Enough to show that φ_y is an extreme point of $K(B)$. If so, by the definition of the Choquet boundary, y is in ∂B and this completes the proof. We must show that φ_y is an extreme point of $K(B)$. Suppose that $\varphi_y \pm U \in K(B)$ for U in B^*. It is easily verified that $(\varphi_y \pm U) \circ T = \varphi_y \circ T \pm U \circ T \in K(A)$. From the definition of the extreme point and Lemma 2, $U \circ T = 0$. This means that $U(Tf) = 0$ for all f in A. Hence $U=0$.

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