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On R_1 spaces

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0. Introduction

R_1 spaces were introduced in 1961 by A.S. Davis [1]. In 1975, W. Dunham [2] investigated R_1 spaces and proved that R_1 spaces, which are weaker than T_2 and stronger than T_1, preserve many of the properties of T_2 and regular spaces. Applying induced maps and natural maps, C. Dorsett [3] further investigated T_0-idenification spaces and the results obtained many properties of R_1 spaces in 1978.

In this paper, we shall give additional properties of R_1 spaces and a generalization of a result proved by C. Dorsett[3]: If f is a continuous function from a compact R_1 space $(X, \mathcal{T})$ onto a T_0 space $(Y, \mathcal{L})$, then $(Y, \mathcal{L})$ is T_2 if and only if f is closed.

1. Definitions and Preliminaries

We employ the following definitions of [1], [3], [5] and [7].

Definition 1.1. A space $(X, \mathcal{T})$ is R_1 iff for each points x_1 and x_2 in X such that $\overline{\{x_1\}} \neq \overline{\{x_2\}}$, there exists disjoint open sets U_1 and U_2 such that $\overline{\{x_1\}} \subset U_1$ and $\overline{\{x_2\}} \subset U_2$.

Definition 1.2. Let R be the equivalence relation on a space $(X, \mathcal{T})$ defined by $x_1 R x_2$ iff $\overline{\{x_1\}} = \overline{\{x_2\}}$. Then the *$T_0$-identification space* of $(X, \mathcal{T})$ is $(X^*, \mathcal{T}^*)$, where X^* is the set of equivalence classes of R and $\mathcal{T}^*$ is the decomposition topology on X^*, which is T_0.

Definition 1.3. A function $f: (X, \mathcal{T}) \to (Y, \mathcal{L})$ is said to be *strongly θ-continuous* if for each point x in X and each open set $V \in \mathcal{L}$ containing $f(x)$, there exists an open set $U \in \mathcal{T}$ containing x such that $f(\bar{U}) \subset V$.

Definition 1.4. If f is a function from a space $(X, \mathcal{T})$ onto a space $(Y, \mathcal{L})$, then the function $f^*: (X^*, \mathcal{T}^*) \to (Y^*, \mathcal{L}^*)$ defined by $f^*(c(x)) = c(f(x))$ is the *induced map from $(X^*, \mathcal{T}^*)$ onto $(Y^*, \mathcal{L}^*)$ determined of f.*

In [2], [4] and [5], W. Dunham, C. Dorsett, P.E. Long and L.L. Herrington proved the following theorems.

Theorem 1.5. *A space $(X, \mathcal{T})$ is T_2 if and only if it is T_0 and R_1.*

Theorem 1.6. *The natural map $P_X: (X, \mathcal{T}) \to (X^*, \mathcal{T}^*)$ is continuous, closed, open, onto and $P_X^{-1}(P_X(U)) = U$ for all $U \in \mathcal{T}$.*

Theorem 1.7. *Let f be a function from a space $(X, \mathcal{T})$ to a space $(Y, \mathcal{L})$. Then the following are equivalent:*

i) *f is strongly θ-continuous.*

ii) *The inverse image of a closed set is θ-closed.*

iii) *The inverse image of an open set is θ-open.*

iv) *For each* $x \in X$ *and each net* $x_\alpha \underset{\theta}{\rightarrow} x$, *then the net* $f(x_\alpha) \rightarrow f(x)$

2. The Main Theorems

Lemma 2.1. *If* f *is a strongly* θ*-continuous function from a space* $(X, \mathcal{T})$ *onto a space* $(Y, \mathcal{L})$, *then* f^* *is a strongly* θ*-continuous function from* $(X^*, \mathcal{T}^*)$ *onto* $(Y^*, \mathcal{L}^*)$.

Proof. Suppose that f is an onto function. Then f^* is an onto function. Let $x \in X$ and $V \in \mathcal{L}$ be an open set containing $f(x)$. Then since f is strongly θ-continuous, there exists an open set $U \in \mathcal{T}$ containing x such that $f(\bar{U}) \subset V$. By Theorem 1.6, put $\bar{U} = P_X^{-1}(\bar{U}^*)$ for all $U^* \in \mathcal{T}^*$. Then $f(P_X^{-1}(\bar{U}^*)) \subset V$. Therefore $P_Y(f(P_X^{-1}(\bar{U}^*)) \subset P_Y(V)$. Thus $f^*(\bar{U}^*) \subset P_Y(V)$. Hence f^* is strongly θ-continuous.

Now, we show additional properties of R_1 spaces by apply Lemma 2.1.

Theorem 2.2. *Let* f *be a strongly* θ*-continuous function from a space*$(X, \mathcal{T})$ *onto a* R_1 *space* $(Y, \mathcal{L})$. *Then* $\{(x_1, x_2) \mid \overline{\{f(x_1)\}} = \overline{\{f(x_2)\}}\}$ *is* θ*-closed in* $X \times X$.

Proof. Suppose that $(Y, \mathcal{L})$ is a R_1 space and $c(y_1) \neq c(y_2)$ for each $c(y_1)$ and $c(y_2)$ in Y^*. Then there exists disjoint open sets V^* and W^* such that $c(y_1) \in V^*$ and $c(y_2) \in W^*$. Therefore $(Y^*, \mathcal{L}^*)$ is a T_2 space. By Lemma 2.1, f^* is strongly θ-continuous since f is strongly θ-continuous. Thus f^* is a strongly θ-continuous function from a space $(X^*, \mathcal{T}^*)$ onto a T_2 space $(Y^*, \mathcal{L}^*)$. Therefore by Theorem 1.7, $\{(c(x_1), c(x_2)) \mid f^*(c(x_1)) = f^*(c(x_2))\}$ is θ-closed in $X^* \times X^*$. Hence $\{(x_1, x_2) \mid \overline{\{f(x_1)\}} = \overline{\{f(x_2)\}}\}$ is θ-closed in $X \times X$.

Corollary 2.3. *Let* f *and* g *be strongly* θ*-continuous functions from a space* $(X, \mathcal{T})$ *onto a* R_1 *space* $(Y, \mathcal{L})$. *Then* $\{x \mid \overline{\{f(x)\}} = \overline{\{g(x)\}}\}$ *is* θ*-closed in* X.

Tneorem 2.4. *Let* f *and* g *be continuous functions from a space* $(X, \mathcal{T})$ *onto a* R_1 *space* $(Y, \mathcal{L})$. *Then* $\{x \mid \overline{\{f(x)\}} = \overline{\{g(x)\}}\}$ *is closed in* X.

Proof. By Theorem 2.2 and [3], it is obvious.

We also obtain the following Theorem 2.5 which is a generalization of a result proved by C. Dorsett[3].

Theorem 2.5. *If* f *is a continuous closed function from a paracompact space* $(X, \mathcal{T})$ *onto a space* $(Y, \mathcal{L})$, *then* $(Y, \mathcal{L})$ *is a* R_1 *space.*

Proof. Suppose that $(X, \mathcal{T})$ is a paracompact space and let $\{V_i \mid i \in I\}$ be any open covering of Y. Then $\{V_i \mid i \in I\}$ has an open refinement which can be decomposed into at most countably many nbd-finite families since f is continuous, closed and onto. Thus $(Y, \mathcal{L})$ is a paracompact space. Therefore $(Y, \mathcal{L})$ is a weakly T_2 space by [2] and [6]. Let $\overline{\{y_1\}} \neq \overline{\{y_2\}}$ for each points y_1 and y_2 in Y. Then $y_1 \in V \in \mathcal{L}$ and $y_2 \in W \in \mathcal{L}$, where V and W are disjoint open sets. Since $(Y, \mathcal{L})$ is a R_0 space, $\overline{\{y_1\}} \subset V$ and $\overline{\{y_2\}} \subset W$. Therefore $(Y, \mathcal{L})$ is a R_1 space.

The following Corollary 2.6 and 2.7 follow immediately from Theorem 1.5 and 2.5.

Corollary 2.6. *If* f *is a continuous function from a compact space* $(X, \mathcal{T})$, *onto a* T_0 *space* $(Y, \mathcal{L})$, *then* $(Y, \mathcal{L})$ *is* T_2 *if and only if* f *is closed.*

Corollary 2.7. ([3]) *If* f *is a continuous function from a compact* R_1 *space* $(X, \mathcal{T})$ *onto a* T_0 *space*

$(Y, \mathcal{L})$, *then* $(Y, \mathcal{L})$ *is* T_2 *if and only if* f *is closed.*

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