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EQUATION GOVERNING SURFACE WAVES OF SMALL AMPLITUDE IN THE PRESENCE OF ROTATIONAL FLOW

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ABSTRACT

Wind wave generation is generally accompanied by a strong wind drift current near the surface. The current can not be considered irrotational. The classical equation governing free surface wave is formulated on the basis of irrotationality. To deal with the situation properly, an equation governing free surface fluctuation of small amplitude is derived when the mean flow can not be assumed irrotational. The equation encompass the classical one as a limiting case as expected.

INTRODUCTION

When surface waves are analysed, generally the irrotationality is assumed and consequently the motion associated with wave in fluid is simply governed the by Laplace equation (e.g., Phillips(1977); Landau and Lifshitz(1959)). The assumption is justified by the fact that fluid is either at rest or in uniform motion from the beginning when the viscosity is neglected. However, simple observation (Choi (1977)) shows that when wind blows over a calm surface of water, the phenomenon being observed is a strong drift current followed by formation of waves. Generally the magnitude of the current increases with fetch value and decreases with depth. As a good approximation the current may be considered irrotational and the motion related to wave will be irrotational too. For a more basic and fundamental analysis of wave-current interaction, it is desirable to formulate an equation governing surface wave without any condition on the irrotationality. This equation must encompass the classical one as a limiting case.

In this paper, an equation governing small surface wave is obtained from the fundamental equations of fluid dynamics, i.e., equation of continuity and momentum equation. The results are discussed comparing with those known classically as a limiting case.

BASIC EQUATION

The schematic representation of the physical situation is given in Figure 1. Wave motion is assumed to be two dimensional. The x axis is taken in the direction of flow and y axis in the direction normal to x and upwards. Fluid is assumed to be incompressible and have infinite depth. In the following analysis viscous effect will be neglected under the condition that Reynolds number concerned is sufficiently large.

The continuity equation is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{1}$$

and the momentum equations are

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}-g \tag{2}$$

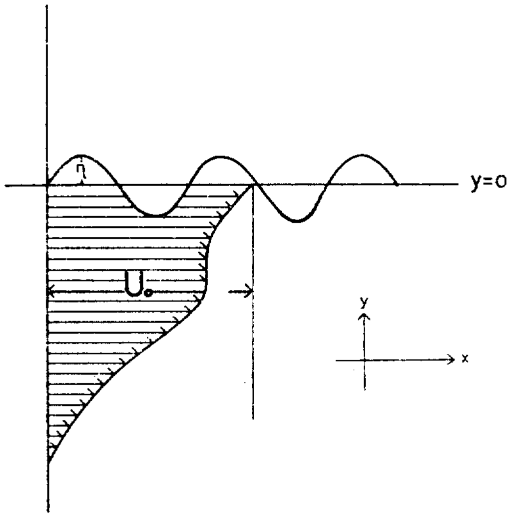

Fig. 1. Schematic representation of the physical situation.

where u and v are x and y components of velocity respectively and p is the pressure and g is gravitational acceleration.

In the absence of wave motion, the undisturbed velocity components and pressure can be taken without loss of generality

$$u=U(y)$$
$$v=0 \qquad (3)$$
$$p=P$$

Equation (3) satisfies equation (1) authomatically and equation (2) becomes

$$\frac{\partial P}{\partial x}=0$$
$$\frac{\partial P}{\partial y}=-\rho g \qquad (4)$$

which gives

$$P=-\rho gy+P_0 \qquad (5)$$

where P_0 is pressure at the surface.

When the surface fluctuation is taken into account, the velocity components and pressure can be put

$$u=U(y)+u'$$
$$v=v' \qquad (6)$$
$$p=P+p'$$

where $'$ denotes fluctuation terms. Substituting equation (6) into equations (1) and (2) yields

$$\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}=0 \qquad (7)$$

$$\frac{\partial u'}{\partial t}+U(y)\frac{\partial u'}{\partial y}+v'\frac{\partial U(y)}{\partial y}$$
$$=-\frac{1}{\rho}\frac{\partial p'}{\partial x} \qquad (8)$$

$$\frac{\partial v'}{\partial t}+U(y)\frac{\partial v'}{\partial y}=-\frac{1}{\rho}\frac{\partial p'}{\partial y} \qquad (9)$$

where the higher order terms are neglected under the condition that the fluctuating terms are small enough compared to the undisturbed term.

Now consider the condition at the boundaries. Condition that motion associated with surface fluctuation must vanish as $y\to-\infty$ can be written

$$v'=0 \quad \text{as} \quad y\to-\infty \qquad (10)$$

Dynamic condition that pressure difference across the free surface is equal to that due to surface tension is

$$[p]=T(R_1^{-1}+R_2^{-1}) \qquad (11)$$

where

$$R_1^{-1}+R_2^{-1}=\frac{\eta_{xx}(1+\eta_y^2)+\eta_{yy}(1+\eta_x^2)-2\eta_{xy}\eta_x\eta_y}{(1+\eta_x^2+\eta_y^2)^{3/2}} \qquad (12)$$

the subscript x and y meaning derivative with respect to x and y respectively, T being the coefficient of surface tension and η being the surface elevation.

The kinematic boundary condition at the surface is

$$\frac{D}{Dt}(\eta-y)=0 \qquad (13)$$

where

$$\frac{D}{Dt}=\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}$$

The conditions (11) and (13) can be linearized on introducing (5) and (6) as

$$\frac{\partial \eta}{\partial t}+U\frac{\partial \eta}{\partial x}=v' \quad \text{at } y=0 \qquad (14)$$

$$\frac{p'}{\rho}+\frac{T}{\rho}\frac{\partial^2 \eta}{\partial x^2}-g\eta=0 \quad \text{at } y=0 \qquad (15)$$

The equations (10), (14) and (15) constitute boundary conditions to be satisfied by surface fluctuation.

EQUATION GOVERNING SURFACE ELEVATION

The classical equation governing the free surface wave is

$$\nabla^2\phi=0 \tag{16}$$

$$\rho g\frac{\partial\phi}{\partial y}+\rho\frac{\partial^2\phi}{\partial t^2}-T\frac{\partial}{\partial y}\left(\frac{\partial^2\phi}{\partial x^2}\right)=0 \quad \text{at } y=0 \tag{17}$$

$$\frac{\partial\phi}{\partial y}\to 0 \quad \text{as} \quad y\to-\infty \tag{18}$$

where ϕ is the velocity potential. The dynamic condition (17) on the free surface is implemented through the combination of kinematic condition and a Bernouille equation which is the integral of momentum equation. The assumption that the motion concerned is irrotational enables to use the Bernouille equation.

In the present analysis, the irrotationality is not assumed a priori and the Bernouille equation can not be used. One of the purposes of present analysis is to see how the rotationality modifies equations (16) and (17). Principally equations (7), (8) and (9) with boundary conditions (10), (14) and (15) can be solved to give the evolution of surface fluctuations as a function of x and t. However, a more tractable and useful form is obtainable.

Equation (7) allows to introduce stream function ψ defined by

$$u'=-\frac{\partial\psi}{\partial y}, \qquad v'=\frac{\partial\psi}{\partial x} \tag{19}$$

In terms of generalized Fourier transform, ψ and η can be expressed

$$\psi=\int\phi_k(y)A_k(t)e^{ikx}dk \tag{20}$$

and

$$\eta=\int a_k(t)e^{ikx}dk. \tag{21}$$

respectively.

In classical analysis, the dynamic boundary condition (17) is obtained from equations (14) and (15) with linearized the Bernouille equation by eliminating pressure term. In the present case, however, the pressure p' at the surface must be derived from either equation (8) or equation (9). Integrating equation (9) from $-\infty$ to 0 yields pressure at $y=0$(Brooke Benjamin(1960)) as

$$-\frac{p'}{\rho}=\int_{-\infty}^{0}\left(\frac{\partial v'}{\partial t}+U(y)\frac{\partial v'}{\partial x}\right)dy \tag{22}$$

Introducing equations (20) and (19) into equation (22) gives

$$-\frac{p'}{\rho}=\int\left[ik\langle\phi_k(y)\rangle\frac{dA_k(t)}{dt}-k^2\langle\phi_k(y)U(y)\rangle A_k(t)\right]e^{ikx}dk \tag{23}$$

where

$$\langle\quad\rangle=\int_{-\infty}^{0}dy$$

Substituting equations (23) and (21) into equation (15) yields

$$-ik\langle\phi_k(y)\rangle\frac{dA_k(t)}{dt}+k^2\langle\phi(y)U(y)\rangle A_k(t)-\left(\frac{T}{\rho}k^2+g\right)a_k(t)=0 \tag{24}$$

while by introducing equations (20), (21) and (19), equation (14) can be written as

$$\int\left(\frac{da_k(t)}{dt}+ik\,U_0a_k(t)-ik\phi_{k0}A_k(t)\right)e^{ikx}dk=0$$

which becomes

$$-ik\phi_{k0}A_k(t)+\frac{da_k(t)}{dt}+ikU_0a_k(t)=0 \tag{25}$$

where 0 denotes that the value is evaluated at $y=0$

In the following analysis, the subscript k and the parenthesis (t) and (y) will be omitted.

Eliminating $A_k(t)$ between equations (24) and (25) gives

$$-\frac{\langle\phi\rangle}{\phi_0}\frac{d}{dt}\left(\frac{da}{dt}+ikU_0a\right)$$
$$-ik\frac{\langle\phi U\rangle}{\phi_0}\left(\frac{da}{dt}+ikU_0a\right)$$
$$-\left(\frac{T}{\rho}k^2+g\right)a=0 \qquad (26)$$

which, by partial integration of $\langle\phi U\rangle$, can be put as

$$\left(\frac{d}{dt}+ikU_0\right)^2 a-ik\frac{\left\langle\frac{dU}{dy}\left\{\int\phi dy\right\}\right\rangle}{\langle\phi\rangle}\cdot$$
$$\left(\frac{d}{dt}+ikU_0\right)a+\frac{\phi_0}{\langle\phi\rangle}\left(\frac{T}{\rho}k^2+g\right)a$$
$$=0 \qquad (27)$$

The boundary condition (10) becomes

$$\phi\to 0 \quad \text{as} \quad y\to-\infty \qquad (28)$$

Eliminating p' in equations (8) and (9) and introducing equations (19), (20) and (21) yield the equation governing $\phi_k(y)$ in the form of the Rayleigh equation

$$(U-C)\left(\frac{d^2}{dy^2}-k^2\right)\phi-\phi\frac{d^2U}{dy^2}=0 \qquad (29)$$

where

$$C=-\frac{1}{ik}\frac{1}{A}\frac{dA}{dt}.$$

Equation (29) with the boundary conditions (27) and (28) substitutes the classical ones (16), (17) and (18).

DISCUSSION

Consider a case where $U=0$, Then equation (29) becomes

$$\left(\frac{d^2}{dy^2}-k^2\right)\phi=0 \qquad (30)$$

whose solution satisfying boundary condition (28) can be put

$$\phi(y)=e^{ky} \qquad (31)$$

Substituting equation (31) into equation (27) gives

$$\frac{d^2a}{dt^2}+\left(\frac{T}{\rho}k^3+gk\right)a=0 \qquad (32)$$

which is an equation governing free surface wave propagation. Equation (32) is similar to that obtained by Phillips(1957) in the analysis of wind wave generation if the pressure fluctuation in the air is neglected. Equation (32) has a solution of the form

$$a(t)=e^{\pm i\sigma t}$$

where

$$\sigma^2=gk+\frac{T}{\rho}k^3$$

Taking into account equation (21), it is easy to show that the phase velocity is

$$c=\pm\sqrt{\frac{g}{k}+\frac{T}{\rho}k} \qquad (33)$$

which is exactly the same one that obtained from classical equation.

Now consider a case where $U=U_0=\text{const.}$ Then equation (29) becomes

$$\left(\frac{d^2}{dy^2}-k^2\right)\phi=0 \qquad (34)$$

and the solution satisfying equation (28) is

$$\phi(y)=e^{ky} \qquad (35)$$

Substituting equation (35) into equation (27) yields

$$\left(\frac{d}{dt}+ikU_0\right)^2 a+\left(\frac{T}{\rho}k^3+gk\right)a=0 \qquad (36)$$

whose solution can be written

$$a(t)=e^{-i\sigma t}$$

where

$$\sigma=-kU_0\pm\sqrt{gk+\frac{T}{\rho}k^3}$$

and then phase velocity is

$$c=U_0\pm\sqrt{\frac{g}{k}+\frac{T}{\rho}k} \qquad (37)$$

which shows that the surface wave considered in the previous case is transported with constant velocity U_0.

Note that one of the undetermined coefficient will be eventually determined by either equation (32) or equation (36) in each case when the initial condition is given. In search for a physical mechanism, Phillips' theory (1957) is based on the solution of equation

(32) taking into account the air pressure fluctuation. On the other hand theory based on the hydrodynamic instability of wind drift current (e.g., Stern and Adam (1973)) begins from equation (29). In instability normal mode analysis makes equation (27) an eigenvalue equation. Both theories actually can not explain satisfactorily the physical mechanism of wind wave generation. It may be valuable to take Phillips' approach beginning from equation (27) when the air pressure fluctuations are considered.

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廻轉流 存在時의 小振幅 表面波의 方程式

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요약 : 現存 小振幅 表面波의 方程式은 風成流가 非廻轉性이라는 假定下에서 유도된 것이다. 그러나 波浪은 바람에 의해 강한 유체의 흐름이 발생된 다음에 형성된다. 이 흐름은 일반적으로 회전성이다. 따라서 廻轉流가 存在할 경우에 小振幅表面波를 지배하는 방정식이 필요하다. 본 연구결과로 얻은 方程式은 非廻轉性의 경우도 극한 값으로 포함한다.