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Structural Aspects in the Theory of Random Walk

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Random walks as special Markov stochastic processes have received particular attention in recent years. Not only the applicability of the theory already developed but also its extension within the frame work of probability measures on algebraic-topological structures such as semigroups, groups and linear spaces became a new challenge for research work in the field. At the same time new insights into classical problems were obtained which in various cases lead to a more efficient presentation of the subject. Consequently the teaching of random walks at all levels should profit from the recent development.

It has been said that Nicolas Bourbaki, the anonymous founder of the encyclopedia "Eléments de Mathématique", started his remarkable work with a discussion between A. Weil und J. Delsarte about how to teach analysis properly. We are certainly following a well established tradition if we try to describe certain aspects of the theory of random walks on the basis of the new knowledge available with the aim to teach the subject with greater satisfaction and effect.

We chose random walks in order to exemplify our thoughts by two reasons: First of all these stochastic processes are easily motivated and favorably described at various levels of mathematical background without too much technical display. Secondly random walks can be viewed as auxiliary objects in order to establish (approximate) the well known Brownian motion process. Thus an introduction to probabilty theory could well be oriented at A. Joffe's Lecture Notes carrying the title "Promenades aléatoires et mouvement Brownien"(1965). Since an analysis of Brownian motion at less advanced levels of teaching is hardly possible, we restrict ourselves to the discussion of random walks and leave the transition to Brownian motion to the initiative of the reader. Within

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the theory of random walk which has been layed out in selfcontained form in the book (1964) by F. Spitzer we choose the problem of recurrence to describe the structural aspects of the theory. Here recurrence means the capacity of the random walk to return to a given position within a certain interval of time. We intend to discuss recurrence at four levels of increasing mathematical sophistication:

(a) via difference equations,

(b) operator theoretically

(c) probabilistically, and

(d) within the frame work of topological groups.

Thus random walks are regarded here not only as objects of probability theory but at the same time as objects of analysis, with their natural relationship to combinatorics and group theory. At the most advanced level descibed under (d) the significance of the algebraic-topological structure of a random walk will become clear. Various results on random walks depend on the structure of the state space as a locally compact group. This view has become popular since the monograph(1975) of D. Revuz and (1977) of the author. We restrict our attention to the presentation of a small selection of contributions taken from the works of P. Baldi(1981), A. Brunel(1974), Y. Guivarc'h, M. Keane, B. Roynette(1977), and B. Roynette(1978). Despite of these more advanced sources the new insights into a structural understanding of recurrent random walks can already be traced in the by now classical literature. K.L. Chung in his textbook(1974) tries to emphasize the significance of the algebraic properties of the state space of a Markov chain, the book(1960) by J.G. Kemeny and E.J. Snell contains an elaborate treatment of finite Makov chains in terms of their transition matrices, and last not least W. Feller's book(1957) still remains the most comprehensive and elegant source also for those readers who are interested in a complete description of the class of all locally compact groups on which every random walk is recurrent.

1. Bernoulli Random Walks

Let a particle move step by step on the real line $\boldsymbol{R}$. With every step it moves by one unit either to the right or to the left, with probability p or q, $q=1-p$, $p\in[0,1]$, respectively. Without loss of generality we assume that the following step is taken after a unit of time. Then the n-th step is taken at time n. We also assume that the possible positions of the particle are elements of the *lattice* $\boldsymbol{Z}$ of *integers*. Thus we are concerned with a *stochastic process* with $\boldsymbol{Z}$ as its state space, which is also called a *random walk* on $\boldsymbol{Z}$. This random walk can be illustrated by its *path mapping* $n \to S_n$ which assigns to every time n the position S_n of the particle. The following **mathematical description** seems to be in order:

Let X_n denote the *n-th* step (or *n-th* move or *n-th* jump) such that

(a) $X_n = +1$ or $X_n = -1$ with probability p or q respectively.

(*b*) $(X_n)_{n \in N}$ is an independent sequence.

Then the position of the particle at time n (after n steps) is just

$$S_n = S_0 + X_1 + \cdots + X_n.$$

Thus the random walk under discussion is represented as the sequence $(S_n)_{n \in Z_+}$. On the other hand

$$S_n - S_0 = X_1 + \cdots + X_n$$

defines a sum of independent *Bernoulli random variables* such that the random walk $(S_n)_{n \in Z_+}$ is reasonably well named a *Bernoulli random walk.*

A **few typical problems** on a Bernoulli random walk are designed to motivate its *recurrence behavior* which we will make precise in the sequel.

A. Let $a, b \in \boldsymbol{R}$, $a, b \geqq 1$, $c = a + b$. Consider the interval $I = [o, c]$. We assume that the Bernoulli random walk starts at $a \in I$.

Problem. What is the probability that the random walk

(i) hits o before it hits c or that

(ii) it hits c before it hits o?

Solution to (i). For every $1 \leqq j \leqq c-1$ we put

u_j = probability that the random walk
hits o prior to c if it starts in j.

What we have to determine is u_a. For this we consider the *difference equation*

$$u_j = p u_{j+1} + q u_{j-1} \quad (1 \leqq j \leqq c-1) \tag{1}$$

under

$$\begin{cases} u_o = 1 \text{ and} \\ u_c = o. \end{cases} \tag{2}$$

This set up is appropriate. In fact, let the particle be in j. After the next step it will be in $j+1$ with probability p, and under this conditional probability that the particle reaches o prior to c equals u_{j+1}. Moreover, with probability q the particle will be in

$j-1$, and under this condition the corresponding conditional probability is u_{j-1}. The formula of total probability yields equation (1).

Result. For $o \leqq j \leqq c$ we get

$$u_j = \begin{cases} \dfrac{r^j - r^c}{1-r^c}, & \text{if} \quad r = \dfrac{q}{p} \neq 1, \\ \dfrac{c-j}{c}, & \text{if} \quad r = 1, \end{cases}$$

and consequently $u_a = \frac{b}{c}$ whenever $r=1$.

Solution to (ii). This is achieved via the defintion

v_j=probability that the random walk hits c prior to o if it starts in j.

Result. For $o \leqq j \leqq c$ we get

$$v_j = \begin{cases} \dfrac{1-r^j}{1-r^c}, & \text{if} \quad r \neq 1, \\ \dfrac{j}{c}, & \text{if} \quad r = 1, \end{cases}$$

which implies $v_a = \frac{a}{c}$ whenever $r=1$.

In general we observe that $u_j + v_j = 1$ for all $o \leqq j \leqq c$, a fact which leads us to the following

B. **Problem.** Let the Bernoulli random walk start in I. What is the probability that it ever reaches the boundary of I?

Result. The random walk does not remain with probability 1 in the given interval I.

This statement motivates the notion of *waiting time* W_j *for the interval* I *and start in* j defined as the first time at which the random walk hits o before c after it has started in j. From the preceding discussion we know that W_j takes values in $\boldsymbol{Z}_+$.

For the expected value $e_j = E(W_j)$ of the waiting time W_j we obtain a difference equation analogous to the above one

(1′) $$e_j = pe_{j+1} + qe_{j-1} + 1 \quad (1 \leqq j \leqq c-1)$$

under

(2′) $$\begin{cases} e_o = 0 \text{ and} \\ e_c = 0. \end{cases}$$

If $p = q = \frac{1}{2}$, its solution is $e_j = j(c-j)$.

We return to Problem B and look at the *limiting behavior* of $u_j = u_j(c)$ for $c \to \infty$ (Open boundary).

Clearly

$$\lim_{c\to\infty} u_j(c)=\begin{cases} r^j, & \text{if} \quad r<1, \\ 1, & \text{if} \quad r\geqq 1. \end{cases}$$

C. **Problem.** Let the Bernoulli random walk start in $a>1$. What is the probability that it will eventually return (recur) to a?

Result. This probability obviously equals

$$\begin{cases} \left(\frac{q}{p}\right)^a, & \text{if} \quad p>q, \\ 1, & \text{if} \quad p\leqq q. \end{cases}$$

In the special case of a *symmetric* Bernoulli random walk defined by $p=q=\frac{1}{2}$, any point is hit arbitrarily often with probability 1 wherever the random walk starts, i.e.. Symmetric Bernoulli random walks are *recurrent.*

We add a **game theoretic interpretation** of the above problems A, B and C. This interpretation concerns the well known *ruin problem* for gamblers. Let G_I and G_{II} be two gamblers gaining with probabilities p and q respectively. It is the rule that in each of the independent parts the loser pays 1 dollar say to the winner. G_I and G_{II} start with initial capitals of a or b dollars, respectively.

A G **Problem.** What is the probability that either G_I or G_{II} is ruined?

Result in terms of the results to A(i) and A(ii):

G_I is ruined$\Longleftrightarrow$The Bernoulli random walk reaches o before c.

G_{II} is ruined$\Longleftrightarrow$The Bernoulli random walk reaches c before o.

B G **Problem.** What is the probability that either G_I or G_{II} will be ruined?

Result. This probability is 1, i.e., eventually G_I or G_{II} will be ruined (if there is no time limit to the game!)

The notions of waiting (or absorption) time refer to the *length of the game.*

Obviously

$\lim_{c\to\infty} u_j(c)=$*probability of ruin* of G_I in a game against the infinitely rich partner G_{II}.

C G **Problem.** We discuss the probability of ruin in the context of the result to Problem C.

The case $p>q$, i.e., G_I has better chances than G_{II}. Let G_I possess just 1 dollar, and let G_{II} be infinitely rich. Nevertheless G_I has a chance $1-\frac{q}{p}$ of avoiding the ruin. Thus in **the case** $p=q$, G_I can eventually win with probability 1 any prescribed amount. This statement, however, takes into account the fact that arbitrarily large debts might become necessary during the game. Under the assumption $p=q$ the game is called *fair*.

2. The Analytic Approach

We shall now consider the *p-dimensional lattice* $\boldsymbol{Z}^d$ as a discrete subgroup of the p-dimensional Euclidean space R^d furnished with the usual norm $\|\cdot\|$ defined by

$$\|x\|=\left(\sum_{k=1}^{d}|x^k|^2\right)^{\frac{1}{2}}$$

for all $x=(x^1, \cdots, x^d)\in R^d$ $(d\geqq 1)$.

Definition. A *transition function* P on $E=\boldsymbol{Z}^d$ is a mapping $E\times E\to R$ having the following properties:

(T1) (Positivity) $P(x, y)\geqq 0$ for all $x, y\in E$.

(T2) (Spatial homogeneity) $P(x, y)=P(o, y-x)$ for all $x, y\in E$.

(T3) (Norming) $\sum_{x\in E}P(o, x)=1$.

(T2) implies that P is determined by the function $x\to p(x)=P(o, x)$ on E satisfying

(T1′) $p(x)\geqq 0$ for all $x\in E$.

(T3′) $\sum_{x\in E}p(x)=1$ for all $x\in E$.

Thus p can be interpreted as a probability measure on E.

We shall use the **terminology** that P describes a random walk on the space $E=Z^d$, or that *P is a riandom walk on the d-dimensional lattice.*

Examples. (1) The Bernoulli random walk $(d=1)$ is given by

$$P(o, x)=\begin{cases} p\geqq 0, & \text{if } x=1,\\ q=1-p\geqq 0, & \text{if } x=-1. \end{cases}$$

(2) The *simple random walk* $(d\geqq 1)$ is defined by

$$P(o, x)=\begin{cases} \frac{1}{2d}, & \text{if } \|x\|=1,\\ 0, & \text{if } \|x\|\neq 1, \end{cases}$$

For $d=1$ we obtain the *symmetric Bernoulli random walk.*

Given an arbitrary random walk P on $E=\boldsymbol{Z}^d$ we note that $P(o, x)$ equals the probability of a one-step transition from o to x. We want to study the probability $P_n(o, x)$ that the random walk starting at o arrives at $x\in E$ after an n-step transition described by P. In Example (1) we have

$$P_n(o, x) = \text{probability of } \frac{n+x}{2} \text{ successes in } n \text{ independent Bernoulli trials}$$

$$= \begin{cases} p^{\frac{n+x}{2}} q^{\frac{n-x}{2}} \binom{n}{\frac{n+x}{2}} & \text{if } n+x \text{ is even, } |x| \leqq n, \\ o, & \text{otherwise.} \end{cases}$$

More generally we define for the given random walk P on E

$$P_o(x, y) = \varepsilon(x, y) = \begin{cases} 1, & \text{if } x = y, \\ 0, & \text{if } x \neq y, \end{cases}$$

$$P_1(x, y) = P(x, y) \text{ for all } (x, y) \in E \times E,$$

and for $n \geqq 2$

$$P_n(x, y) = \sum_{(x_1, \ldots, x_{n-1}) \in E^{n-1}} P(x, x_1) P(x_1, x_2) \cdot \ldots \cdot P(x_{n-1}, y)$$

whatever $(x, y) \in E \times E$.

We note that for every pair $(x, y) \in E \times E$

$P_n(x, y)$ = probability that the random walk starting in x at time o will reach y at time n.

Here we do not restrict the random walk's positions between times o and n. But this observation gives rise to the following

Definition. For $n \geqq 0$ let

$F_n(x, y)$ = probability that the random walk starting in x at time o will reach y at time n *for the first time.*

More formally we put

$$F_0(x, y) = 0 \text{ for all } (x, y) \in E \times E,$$

$$F_1(x, y) = P(x, y) \text{ for all } (x, y) \in E \times E,$$

and for $n \geqq 2$

$$F_n(x, y) = \sum_{(x_1, \ldots, x_{n-1}) \in (E \setminus \{y\})^{n-1}} P(x, x_1) P(x_1, x_2) \cdot \ldots \cdot P(x_{n-1}, y)$$

whatever $(x, y) \in E \times E$.

More notation. For all $n \in \boldsymbol{Z}_+$, $x, y \in E$ we put

$$G_n(x, y) = \sum_{k=0}^{n} P_k(x, y),$$

$$G(x, y) = \lim_{n \to \infty} G_n(x, y) = \sum_{k \geqq 0} P_k(x, y) \leqq \infty,$$

and

$$G = G(0, 0).$$

Analogously we set

$$F(x, y)=\lim_{n\to\infty} F_n(x, y)\leqq 1,$$

and

$$F=F(0, 0).$$

Definition. A random walk P on E is said to be recurrent if $F=1$.

The following result is easy to prove:

Theorem. $G=\frac{1}{1-F}$
with the conventions $G=\infty$ iff $F=1$.

Example (1). For the Bernoulli random walk introduced above one computes for all $n\geqq 1$

$$P_n(0, 0)=0 \text{ if } n \text{ is odd}$$

and

$$P_{2n}(0, 0)=(pq)^n\binom{2n}{n}=(-1)^n(4pq)^n\binom{-\frac{1}{2}}{n},$$

hence obtains

$$G=\begin{cases}(1-4pq)^{-\frac{1}{2}}<\infty, & \text{if} \quad p\neq q,\\ \infty, & \text{if} \quad p=q.\end{cases}$$

Résumé. The Bernoulli random walk on $\boldsymbol{Z}$ is recurrent iff $p=q=\frac{1}{2}$, i.e., on $\boldsymbol{Z}$ the symmetric Bernoulli random walk is the only recurrent simple random walk.

So far we obtained a recurrence result only in the case of dimension $d=1$. For dimension $d\geqq 2$ the situation is more complicated as Polyá observed already in 1921.

3. The Probabilistic Elaboration

From now on the measure theoretic foundation of probability theory will be used in order to handle the notions of a probability space, random variables, their distributions, the expected value, stochastic independence and so forth.

The probabilistic description of a random walk P on $E=\boldsymbol{Z}^d$ for $d\geqq 1$ can be given in terms of its set of paths or in terms of its sequence of jumps. In any case one constructs for the given transition function P on E a probability space

$$(\Omega, \mathscr{A}, Pr)=(E, \mathscr{P}(E), P)^N$$

where $\Omega=E^N$, $\mathscr{A}=\sigma$-algebra generated by the cylinder sets $Z_{A,n}$ of the form

$$Z_{A,n}=\{\omega=(\omega_n)_{n\in N}\in\Omega : (\omega_1,\ldots,\omega_n)\in A\}$$

for sets $A\subset E^n$, $n\in N$, and Pr is a probability measure on $(\Omega,\mathscr{A})$ determined by

$$Pr[X_1=x_1,\ \cdots,\ X_n=x_n]=P(o,x_1)\cdot\ldots\cdot P(o,x_n)$$

for all $x_1,\ \cdots,\ x_n\in E,\ n\in N.$

For every $k\geqq 1$ we denote by X_k the k-th projection of Ω onto E, and for every $n\geqq 1$ we set

$$\begin{cases} S_o=0 \text{ as well as,} \\ S_n=\sum_{k=1}^{n} X_k. \end{cases}$$

Clearly for every $n\geqq 1$ and $x,y\in E$ we have the equality

$$P_n(x,\ y)=Pr[S_n=y-x]$$

which justifies the previously suggested interpretation of the transitions of the random walk P. For any $\omega\in\Omega$

$(S_1(\omega), S_2(\omega),\ldots)$ describes the *path* of P and

$(X_1(\omega), X_2(\omega),\ldots)$ its *sequence of jumps.*

Definition. A mapping T from the measurable space $(\Omega,\mathscr{A})$ into the measurable space $(N\cup\{\infty\},\ \mathscr{P}(N\cup\{\infty\}))$ is said to be a stopping time for the random walk P if for all $n\geqq 1$

$$[T=n]\in\mathscr{A}([X_k=x_k] : x_k\in E \text{ for } k=1,\cdots,n),$$

where the latter symbol stands for the σ-algebra generated by the sets $[X_k=x_k]$ for $x_k\in E,\ k=1,\ \cdots,\ n.$

We note that the events $[T=n]$ depend only on the *past* $\{X_1,\cdots,X_n\}$, but not on the *future* $\{X_{n+1},X_{n+2},\cdots\}$ of the random walk $P.$

Example. Let $A\subset E.$ We introduce by

$$T_A(\omega)=\begin{cases}\inf\ \{n\in N: S_n(\omega)\in A\} & \text{if}\,\{\ldots\}\neq\phi\ , \\ \infty & \text{otherwise}\ ,\end{cases}$$

for all $\omega\in\Omega$ the *first hitting time* of the set A.

Obviously T_A is a stopping time for P.

With this terminology we can continue our probabilistic interpretation.

For every $n\in N$ and $x,y\in E,$

$$F_n(x,y)=Pr[T_{\{y-x\}}=n],$$

for $x,\ y\in E$

$$F(x,y)=Pr[T_{\{y-x\}}<\infty].$$

Now let us introduce the measurable sets

$$A_k=[S_k=0] \text{ for } k\geqq 1$$

and

$$A_\infty=\overline{\lim_{n\to\infty}}\, A_n=\bigcap_{n\geqq 1}\bigcup_{k\geqq n} A_k.$$

Theorem. Let P be a random walk on $E:=\boldsymbol{Z}^d$ for $d\geqq 1$.

Then

(i) P is recurrent iff $Pr(A_\infty)=1$.

(ii) P is transient iff $Pr(S_\infty)=0$.

Further interpretations can be given. Since for $n\in N$ and $x\in E$

$$P_n(o,x)=Pr[S_n=x]=E_{Pr}(1_{[S_n=x]}),$$

$$G_n(o,x)=E_{Pr}\left(\sum_{k=0}^{n}1_{[S_k=x]}\right)$$

equals the *expected number of visits* in $x\in E$ within n units of time when the random walk starts in o.

4. The Structure of Recurrent Random Walks

Let G be a locally compact group (written multiplicatively with unit e) having a countable basis of its topology. We consider the set $\mathcal{M}^1(G)=\mathcal{M}^1(G,\mathcal{B})$ of all Radon (or Borel) probability measures on (the Borel σ-algebra $\mathcal{B}$ of) G. In $\mathcal{M}^1(G)$ we introduce the vague topology and the *convolution operation* defined for the measures $\mu,\nu\in\mathcal{M}^1(G)$ by

$$\mu*\nu(f)=\iint f(xy)\mu(dx)\nu(dy)$$

for all continuous real functions real functions f on G with compact support. Furnished with vague topology and convolution $\mathcal{M}^1(G)$ becomes a topological semigroup with the Dirac measure ε_e in e as neutral element. $\mathcal{M}^1(G)$ is commutative iff G is Abelian, and compact iff G is compact. Further properties of $\mathcal{M}^1(G)$ are discussed in the literature *on probability theory on locally compact groups.*

Definition. A *transition function* (or Markov kernel) on G is a mapping $P: G\times\mathcal{B}\to R$ such that

(M1) $P(x,B)\geqq 0$ for all $(x,B)\in G\times\mathcal{B}$.

(M2) For all $B\in\mathcal{B}$ the mapping $x\to P(x,B)$ is $\mathcal{B}$-measurable.

(M3) For all $x\in G$ the mapping $B\to P(x,B)$ is a probability measure on $(G,\mathcal{B})$.

A transition function P on G is said to be *translation invariant* if

$$P(x+y, B+y)=P(x, B)$$

for all $(x, B)\in G\times\mathscr{B}$ and $y\in G$.

For any measure $\mu\in\mathcal{M}^1(G)$ we define a *random walk on G* by the transition function P_μ on G given by

$$P_\mu(x, B)=(\varepsilon_x*\mu)(B)=\mu(Bx^{-1})$$

for all $(x, B)\in G\times\mathscr{B}$.

Examples. The random walks discussed in the previous sections are, of course, random walks on the group $G=\boldsymbol{Z}^d$ for $d\geqq 1$. Since in this case G is discrete, it suffices to consider the transition probabilities

$$P(x, y)=P_\mu(x, \{y\})=\mu(y-x)$$

for all $x, y\in G$, where μ denotes the probability measure on G determined by P in the form

$$\mu(\{x\})=P(o, x)$$

for all $x\in G$.

There is a probabilistic description of the random walk P_μ on G as in Section 3, i.e., for P_μ there exists a *stochastic process* (a Markov chain) to be *constructed on the space*

$$(\Omega, \mathscr{A}, Pr)=(G, \mathscr{B}, \mu)^N.$$

Definition. For the random walk P_μ on G we introduce its *potential kernel* as the measure

$$G=\sum_{n\geqq 0}\mu^n$$

on G (which is not necessarily finite).

Definition. The random walk P_μ is called *recurrent* if

$$G(0)=\infty$$

for all nonempty open sets $0\in\mathscr{B}$.

Definition. A locally compact group G (with a countable basis) is said to be *recurrent* if there exists at least one recurrent random walk on G.

Examples.

(1) All compact groups, in particular the torus groups $\boldsymbol{T}^d$ for $d\geqq 1$, are recurrent.

(2) All groups of the form $G=\boldsymbol{R}^{d_1}\times\boldsymbol{Z}^{d_2}$ with $d_1+d_2\leqq 2$ are recurrent.

Remark. An interesting result of *A.* Brunel and *D.* Revuz states that if on G every symmetric random walk is recurrent, then G is necessarily compact.

(3) Free groups with $\geqq 2$ generators are *not* recurrent.

In the following we quote a few results which indicate the intimate **relationship between the existence of recurrent random walks and the structure of its underlying groups.**

Theorem. 1. Let G be a countable discrete Abelian group.

The following statements are equivalent:

(i) G is recurrent.
(ii) For the rank $r(G)$ of G we have $r(G) \leqq 2$.

Theorem. 2. Let G be an arbitrary Abelian locally compact group with a countable basis. Then

$$G \cong \boldsymbol{R}^n \times H,$$

where H contains an open, compact subgroup K.

We have the equivalence of

(i) G is recurrent.

(ii) $G_{/\{e\} \times k}$ is recurrent.

But $G_{/\{e\} \times k} \cong R^n \times G_1$,

where G_1 is countable.

Then (with Theorem 1) we get the equivalence of

(i) G is recurrent.

(iii) $n + r(G_1) \leqq 2$.

Simple application.

$\boldsymbol{Z}^d$ is recurrent iff $d \leqq 2$.

This statement is related to Polyá's result that the simple random walk is recurrent on $\boldsymbol{Z}^2$, but is transient on $\boldsymbol{Z}^3$.

The most complete result concerning recurrence of groups is due to *P.* Baldi.

Theorem. 3. For any connected Lie group G the following statements are equivalent:

(i) G is recurrent.

(ii) G is of polynomial growth of degree $\leqq 2$, i.e., for any relatively compact neighborhood V of e generating the whole group G there exists a constant $c > 0$ such that

$$\omega(V^n) \leqq cn^2$$

for all $n \in N$, where ω denotes a left Haar measure of G.

There is still the *open problem* of characterizing all recurrent groups among the non-Abelian non-connected ones. The following profound result of P.S. Novikov and S.I. Adyan indicates what kind of obstacles have to be overcome: There exists an infinite group G such that $x^N = e$ for all $x \in G$, where N is a fixed integer >0.

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