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Sway Added Mass of a Rectangular Cylinder in a Restricted Water

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Abstract

In this paper, the sway added mass of a rectangular cylinder in a restricted water is considered by applying Hamilton's principle as the frequency tends to zero. The present method is an extension of Isshiki's method proposed in 1978.

In the present method, it is assumed that the fluid velocity distribution in each subdomain of the fluid can be represented by higher order polynomials while Isshiki assumed linear velocity distribution. The fluid flow is assumed as a rotational motion in the present analysis. However, the results obtained from the present method show good agreement with Bai's numerical results for the case of large clearances between a canal wall and a cylinder.

From Kelvin's minimum energy theorem, we can see that the value of sway added mass obtained from the present method approaches the upper bound. The approximate formula obtained in the present study takes a simple form which consists of the dimensions of the canal and the cylinder.

The present formulae are derived for the cases of a rectangular cylinder swaying at the center of a narrow or wide canal relative to a cylinder, at off-center location in a canal, and in the restricted water with a single wall.

From the results of numerical calculation, it is concluded that the sway added mass in restricted waters is more affected by water depth than clearance between a wall and a cylinder.

Symbols

a, b, c, d, e, B, W	: dimensions of cylinder and canal
ξ	: distance from the center of canal to that of cylinder
η	: distance from a wall to cylinder
$R_1 - R_5$	: 5 subdivisions of fluid domain
S, U, S', U'	: juncture interfaces between subdivisons
X	: sway deflection of cylinder
ζ_1, ζ_5	: deflections of water surface
u, v	: fluid velocities in x-direction and y-direction
T, V	: kinetic energy and potential energy
M	: mass of cylinder
ρ	: water density
M_1	: sway added mass of cylinder
F_x	: sway force
σ	: frequency
g	: gravitational acceleration
μ	: added mass coefficient based on displaced fluid mass
*	: represents linearization

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1. Introduction

The maneuvering characteristics of a ship are very important when she passes through a canal or makes a port in accordance with the increasing tendency of a ship size. To know this characteristics, the added mass of a ship in a restricted water has to be calcu lated.

It has been calculated by several methods. Fujino (1973, 1975, 1976) calculated the upper and lower bounds of the added mass for several types of a cylinder using a hyper-circle method. Newman(1976) proposed the approximate formula for calculating the added mass of a rectangular cylinder for the limiting frequency $\sigma\to 0$ in a canal in the discussion of the above paper by Fujino. For the case of small clearance bewteen a canal and a body he derived a formula of sway added mass using simple solutions of Laplace equation.

Meanwhile Bai (1977a) derived the better approximate formula calculating the added mass of a rectangular cylinder for the limiting frequency $\sigma\to 0$ by subdividing the whole fluid domain by 3 subdomains and applying the better solutions of Laplace equation to each subdomain under the same assumption as Newman did. And he (1977c) computed the added mass of cylinders for the limiting frequencies in a restricted water using a finite-element method based on the dual-extremum principles. For the case of small clearances between a body and a canal, Hwang and Yoon(1977) derived the approximate formulae for calculating the added masses of a rectangular cylinder swaying ($\sigma\to 0$, $\sigma\to\infty$) or heaving ($\sigma\to\infty$) in a canal using a finite-element method and showed their formulae are the same with those of Bai (1977a). Isshiki (1978) subdivided the fluid domain into 5 subdomains assumed that the fluid velocity in each subdomain was represented by a linear equation, and derived the formula computing the added mass of a rectangular cylinder using Hamilton's principle. His results for the limiting frequencies were consistent with the approximate formulae of Bai(1977a) and Hwang and Yoon (1977). But his results also have significant discrepancies with the accurate results of Bai (1977c) for the case of the large clearance between a body and a canal.

In this paper, we apply the method, with which Isshiki (1978) has analyzed the problem of the surge motion of a freely- floating ship in a dock, to the cases of a cylinder swaying($\sigma\to 0$) at the center of a wide canal, at off-center location, and with a single wall. The results are shown better agreements with those of accurate numerical calculation of the other investigators.

2. Sway Added Mass Formulae derived from Hamilton's Principle

2.1. Derivation of Sway Added Mass

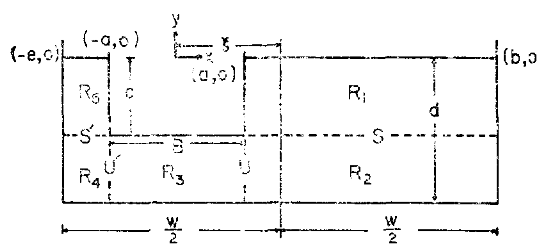

Fig. 1. Coordinate system

The sway added mass formulae of a rectangular cylinder in a restricted water have derived by employing Hamilton's Principle as Isshiki developed in 1978. The fluid region is divided by five subdomains and fluid velocities are assumed appropriately by higher order polynomials in each subdomain for steady state of oscillation.

The kinetic and potential energies of the fluid are calculated in each subdomain. Then, applying Hamilton's Principle to the total system we obtain the added mass formulae of the rectangular cylinder on a free surface in the restricted water.

Finally we obtain the sway added mass formula for a limiting frequency ($\sigma\to 0$) from the above mentioned frequency dependent formula.

The derived formula of the sway added mass coefficient for $\sigma\to 0$ is expressed as

$$\mu_1=\frac{p}{2ac} \tag{2-1}$$

where

$$p=\frac{c(b-a)}{2m+1}+\frac{m^2}{3(2m-1)}\cdot\frac{c^3}{b-a}+\frac{c^2}{2m+1}\cdot\frac{b-a}{d-c}+\frac{m^2}{3(2m-1)}\cdot\frac{c^2(d-c)}{b-a}+\frac{2ac^2}{d-c}+\frac{c^2}{2n+1}\cdot\frac{c-a}{d-c}+\frac{n^2}{3(2n-1)}\cdot\frac{c^2(d-c)}{c-a}+\frac{c}{2n+1}(c-a)+\frac{n^2}{3(2n-1)}\cdot\frac{c^3}{c-a} \tag{2-2}$$

m, n: positive integer

and 2ac is the section area of the rectangular cylinder. The detail derivation of the above formula is shown in appendix.

2.2. In the Case of a Cylinder at the Center of a Canal

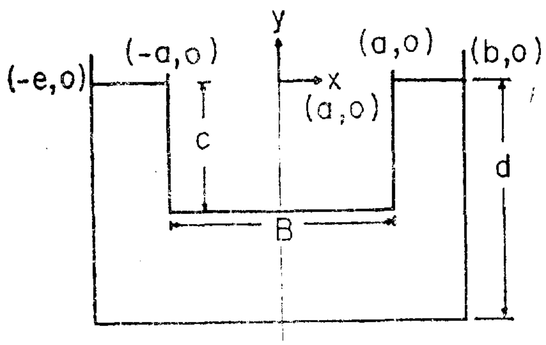

Fig. 2. Swaying rectangular cylinder at the center of a restricted canal

In this case, $\xi=0$, $b=e$ and $m=n$. From eq. (2.1), the equation of the sway added mass coefficient is obtained as follows:

$$\mu_{10}=\frac{1}{a}\left[\frac{1}{2m+1}\cdot\frac{d(b-a)}{d-c}+\frac{m^2}{3(2m-1)}\cdot\frac{cd}{b-a}+\frac{ac}{d-c}\right] \tag{2.3}$$

In the process deriving the above equation, for the fluid velocities in each fluid subdomain irrotationality is not assumed. Then the rotational fluid motion has more kinetic energy than the irrotational fluid motion according to the Kelvin's minimum energy theorem.

Accordingly we must find the most appropriate value of m. Although we can choose the most appropriate value of m according to the ratio of the dimensions of a cylinder and a canal, we will choose only natural numbers as the value of m for convenience. Fig. 3 shows the relation between μ_{10} and variation of m. In this figure a certain value of m has the available region to give the minimum value of μ_{10} for the given dimensions of a cylinder and a canal. We can find the available region of m from eq. (2.3).

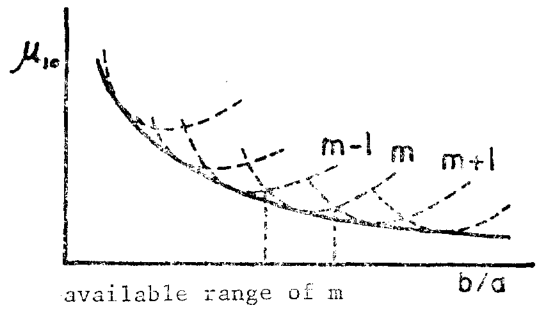

Fig. 3. Relation of μ_{10} and Variation of m

In this sense the following conditions are considered for the selection of m.

$$\frac{p}{2m+1}+\frac{m^2q}{3(2m-1)}\leqq\frac{p}{2(m+1)+1}+\frac{(m+1)^2q}{3(2m+1)} \tag{2.4a}$$

$$\frac{p}{2m+1}+\frac{m^2q}{3(2m-1)}\leqq\frac{p}{2(m-1)+1}+\frac{(m-1)^2q}{3(2m-3)} \tag{2.4b}$$

where $p=\frac{b-a}{d-c}$, $q=\frac{c}{b-a}$, $p>0$ and $q>0$.

The region in which m satisfies (2.4a) and (2.4b) is

$$\frac{\{2(m-1)+3\}\{2(m-1)^2-1\}}{6\{2(m-1)-1\}}\leqq\frac{p}{q}\leqq\frac{(2m^2-1)(2m+3)}{6(2m-1)} \tag{2.5}$$

Especially when $m=1$, the region of p/q is

$$0<\frac{p}{q}\leqq\frac{5}{6} \tag{2.6}$$

The region of p/q for m is tabulated in Table 1.

2.3. In the Case of a Cylinder at Off-Center Location in a Canal

From eq. (2.1), we obtain the equation for calculating μ_1 for the case of a cylinder at off-center location in a canal.

$$\mu_1=\frac{1}{2a}\left[\frac{1}{2m+1}\cdot\frac{d(b-a)}{d-c}+\frac{m^2}{3(2m-1)}\cdot\frac{cd}{b-a}+\frac{2ac}{d-c}\right.$$

$$+\frac{n^2}{3(2n-1)}\cdot\frac{cd}{e-a}+\frac{1}{2n+1}\cdot\frac{d(e-a)}{d-c}\Big] \tag{2.7}$$

By comparing eq. (2.7) with eq. (2.3), we can see that the sway added mass for this case is the mean value of the added mass of left and right side. That is

$$\mu_1=\frac{1}{2}(\mu_{10L}+\mu_{10R}) \tag{2.8}$$

where μ_{10L}: the added mass of left side

μ_{10R}: the added mass of right side

Therefore μ_1 can be obtained from μ_{10}. However, for μ_2 this relation is not hold as we can see from eq. (2.1).

2.4 In the Case of a Cylinder with a Single Wall

This is the extension of the case of the off-center cylinder. For this case, an artificial vertical rigid wall is introduced at a sufficient distance from the body(for example, $\frac{b-a}{a}=100$). Then, the following equation is obtained from eq. (2.7)

$$\mu_1=\frac{1}{d-c}\left(\frac{50d}{2m+1}+c\right)+\frac{m^2cd}{600a^2(2m-1)}$$
$$+\frac{1}{2a}\left[\frac{n^2}{3(2n-1)}\cdot\frac{cd}{\eta}+\frac{1}{2n+1}\cdot\frac{d\eta}{d-c}\right] \tag{2.9}$$

where $\eta=e-a$ and m must satisfy the following condition.

$$\frac{\{2(m-1)+3\}\ \{2(m-1)^2-1\}}{60000\ \{2(m-1)-1\}}\leqq\frac{a^2}{c(d-c)}$$
$$\leqq\frac{(2m^2-1)\ (2m+3)}{60000\ (2m-1)} \tag{2.10}$$

3. Results and Discussion

As mentioned earlier, sway added mass is important to ship motion in a canal or in a restricted water. Especially sway added mass for the limiting frequency $\sigma\to 0$ is very important to ship maneuverability. Equations (2.3), (2.7) and (2.9) give sway added mass of a rectangular cylinder ($\sigma\to 0$) respectively for the case of a cylinder translating at the center of a canal, at the off-center location in a canal, and near a wall.

In the process of deriving these equation, irrotationality of fluid motion is not assumed. By Kelvin'

Table 1. The Range of p/q dependent upon m

m	p/q	m	p/q
1	0～0.83333	21	～161.159
2	～2.72222	22	～176.159
3	～5.10000	23	～191.826
4	～8.11905	24	～208.160
5	～11.7963	25	～225.160
6	～16.1364	24	～242.827
7	～21.1410	27	～261.160
8	～26.8111	28	～280.161
9	～33.1471	29	～299.827
10	～40.1491	30	～320.161
11	～47.8175	31	～341.161
12	～56.1522	32	～362.828
13	～65.1533	33	～385.162
14	～74.8210	34	～408.162
15	～85.1552	35	～431.828
16	～96.1559	36	～456.162
17	～107.823	37	～481.162
18	～120.157	38	～506.829
19	～133.158	39	～533.162
20	～146.825	40	～560.162

Table 2. Sway Added Mass of a Rectangular Cylinder at the Center of a Rectangular Canal ($\sigma\to 0$) $a/c=1.0$

b/a	d/c	Bai (upper bound)	Fujino	Bai (1977a) Isshiki	Present Formula	Relative Error to Bai(%)
1.05	1.5	10.0948	10.090	12.0350	12.0350	19.22
1.1	1.5	6.3709	6.370	7.1000	7.1000	11.44
1.2	1.5	4.4400	4.440	4.7000	4.7000	6.82
1.3	1.5	3.8013	3.805	3.9667	3.9667	4.35
1.5	1.5	3.3393	3.342	3.5000	3.5000	4.81
2.5	1.5	3.0937	3.105	3.8333	3.2429	4.82
3.5	1.5	3.0979	3.112	4.7222	3.2315	4.31
5.0	1.5	3.1114	3.137	6.1217	3.2272	3.72

Table 3. Sway Added Mass of a Rectangular Cylinder at the Off-Center of a Rectangular Canal ($\sigma\to 0$) $a/c=1.0$, $W/a=5.0$

d/c	ξ/B	Bai (upper bound)	Present Formula	Relative Error to Bai (%)
1.05	0.40	22.1658	22.7482	2.63
	0.60	22.2246	22.7640	2.43
	0.65	22.4039	22.9310	2.35
	0.70	23.1119	23.4559	1.49
	0.72	24.1806	24.4827	1.25
	0.74	29.8485	30.1761	1.10
1.5	0.40	3.1571	3.3018	4.58
	0.60	3.4639	3.6024	4.00
	0.65	3.7847	3.9665	4.80
	0.70	4.7579	5.1649	8.55
	0.72	6.0213	6.8118	13.13
	0.74	12.0948	15.1242	25.05

Table 4. Sway Added Mass of a Rectangular Cylinder near a Wall ($\sigma\to 0$). $a/c=1.0$, $d/c=1.05$

η/B	Bai (upper bound)	Present Formula	Relative Error (%)
1.0	22.7466	22.7118	0.15
0.3	22.1825	22.7222	2.43
0.1	22.4160	22.9305	2.30
0.05	23.1182	23.4555	1.46
0.03	24.1816	24.4822	1.24

s minimum energy theorem, the rotational fluid has more kinetic energy than the irrotational fluid as mentioned in the previous section. That is, the added mass in rotational motion is always larger than that in irrotational motion.

When $m=1$, eq. (2.3) is identical with the results of Isshiki and Murakami (1978), Bai (1977a), and Hwang and Yoon (1977). And it may be used only when the clearance between a wall and a cylinder is small, i.e. when it satisfies eq. (2.6). But we can see from Table 2 and Fig. 4 that eq. (2.3) can be used in case of a large clearance by using of m in Table 1.

For the case of the off-center cylinder ($W/a=5$, $d/c=1.05$ and 1.5) we have computed the sway added mass with variation of ξ/B and compared our results with those of Bai (1977c) as shown in Table 3 and Fig. 5, 6. And our results for $W/a=3$, $d/c=1.05$, 1.1 are compared with the results of Fujino (1976) as shown in Fig.7, 8. From these comparisons we can see that the equation for μ_1 of those derived in this report gives good results. However, for the case

Table 5. The Effect of the Side Wall and the Water Depth on the Sway Added Mass of a Ractangular Cylinder in a Canal ($\sigma\to 0$) $a/c=1.0$

d/c	b/a	μ_1
1.3	1.05	9.0722
	1.1	4.8111
	1.2	2.7889
	1.3	2.2111
	1.4	1.9944
	1.5	1.9222
	1.7	1.7654
1.05	1.3	22.8156
1.1		12.3222
1.2		6.9333
1.3		5.2111
1.4		4.4056
1.5		3.9667
1.7		3.5603

Table 6. The Effect of Single Wall and the Water Depth on the Sway Added Mass of a Rectangular Cylinder near a Wall ($\sigma\to 0$). $a/c=1.0$

d/c	η/B	μ_1
1.3	0.025	6.8879
	0.05	4.7574
	0.1	3.7463
	0.15	3.4574
	0.2	3.3491
	0.25	3.3129
	0.35	3.2339
1.05	0.15	22.7638
1.1		12.1657
1.2		6.7413
1.3		4.9574
1.4		4.0918
1.5		3.5957
1.7		3.0810

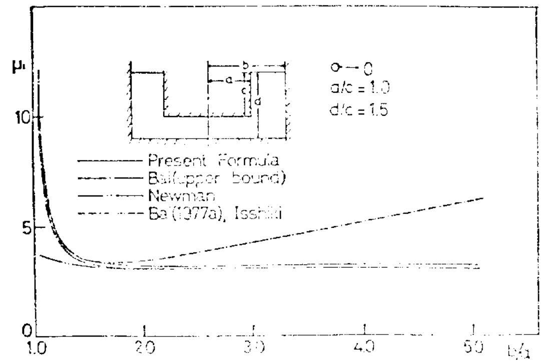

Fig. 4. Sway Added Mass Coefficient of a Rectangular Cylinder at the Center of a Rectangular Canal

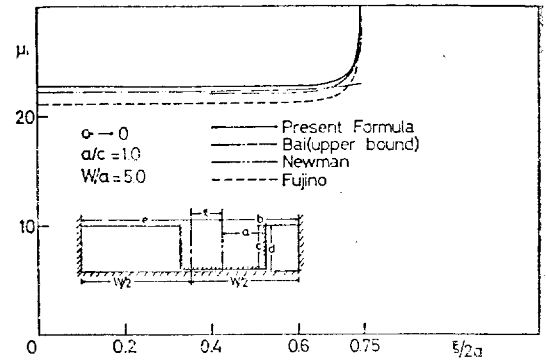

Fig. 5. Sway Added Mass Coefficient of a Rectangular Cylinder at the Off-center of a Rectangular Canal $(d/c=1.05)$

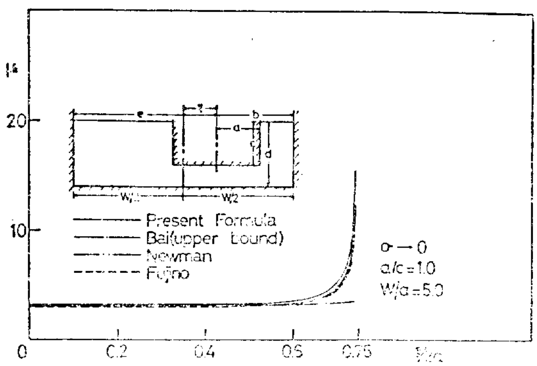

Fig. 6. Sway Added Mass Coefficient of a Rectangular Cylinder at the Off-center of a Rectangular Canal $(d/c=1.5)$

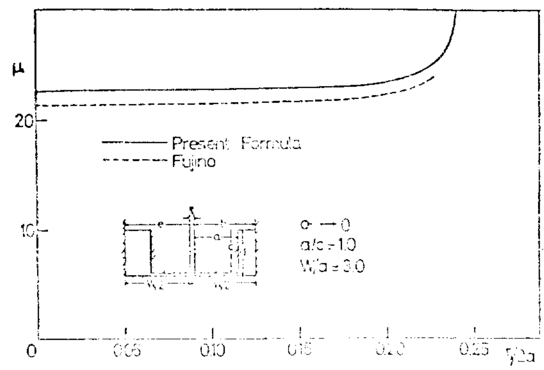

Fig. 7. Sway Added Mass Coefficient of a Rectangular Cylinder at the Off-center of a Rectangular Canal $(d/c=1.05)$

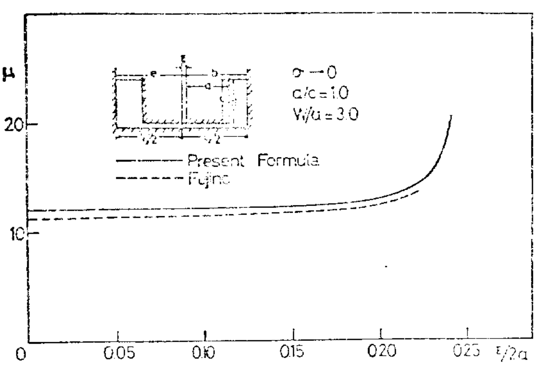

Fig. 8. Sway Added Mass Coefficient of a Rectangular Cylinder at the Off-center of a Rectangular Canal $(d/c=1.1)$

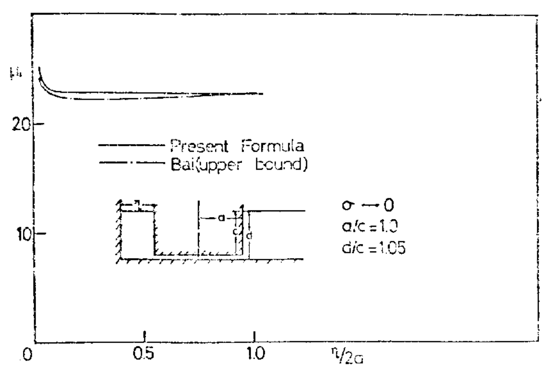

Fig. 9. Sway Added Mass Coefficient of a Rectangular Cylinder near a Wall

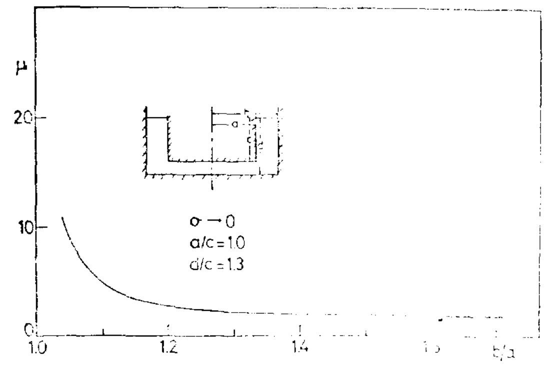

Fig. 10. The Side Wall Effect on the Sway Added Mass of a Rectangular Cylinder in a Rectangular Canal

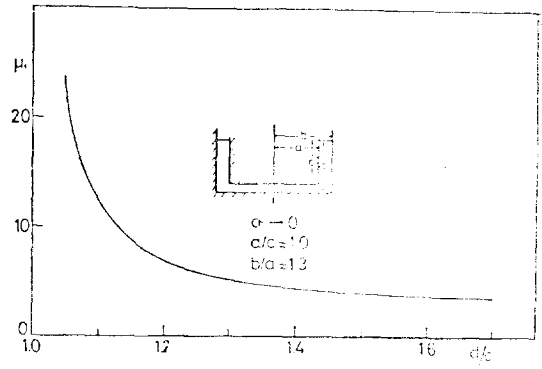

Fig. 11. The Depth Effect on the Sway Added Mass of a Rectangular Cylinder in a Rectangular Canal

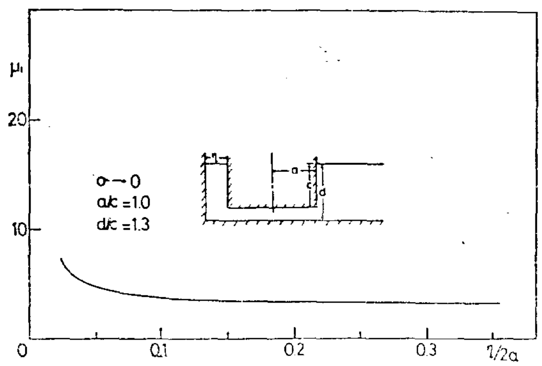

Fig. 12. The Single Wall Effect on the Sway Added Mass of a Rectangular Cylinder

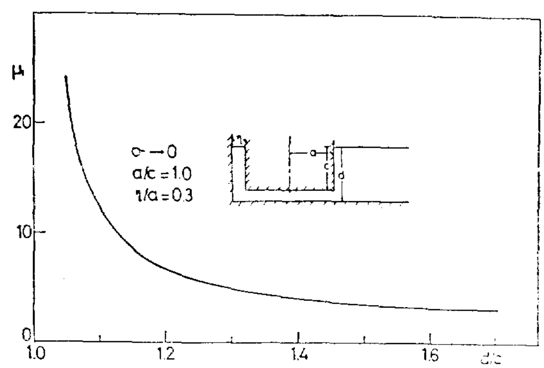

Fig. 13. The Depth Effect on the Sway Added Mass of a Rectangular Cylinder near a Wall

which $d/c=1.5$ and ξ/B is large, the results are insufficient. This seems to be caused by using the assumption that u is independent of y in R_3. If we assume that u is dependent on y in R_3, we may obtain the better results than the above.

The added masses of a cylinder, μ_1, when a single wall is present, are given in Table 4 and Fig.9. These results compared with those of Bai(1977c); it shows that they are in good agreement.

On the other hand μ_2 is poor in accuracy in the region except $0<p/q\leqq 5/6$. This seems to be caused by neglecting the term $\dot{X}$ in the process of linearization.

We have investigated the effect of a bottom and a wall on the added mass. Table 5, Fig. 10 and Fig. 11 show the added mass is affected more by a bottom than a wall. And when the other dimensions are fixed, the added mass scarcely varies in the region $d/c\geqq 1.5$ or $b/a\geqq 1.5$. On the other hand the added mass increases rapidly in the reqion $0<b/a\leqq 1.1$ or $0<d/c\leqq 1.2$.

For the case of a single wall, the effect of a bottom and a wall is similar to the case of both side walls as shown in Table 6, Fig. 12, and Fig. 13.

The method in this report may be applied to the heave motion. Especially the added mass for the limiting frequency($\sigma=0$) which can't be obtained by the other method may be obtained.

4. Conclusion

When a rectangular cylinder translates in a restricted water, we have extended Isshiki-Murakami's method for calculating sway added mass to the case of large clearance between a wall and a cylinder in a limiting case $\sigma \rightarrow 0$ and obtained the following conclusions:

Even the fluid flow is assumed as rotational motion this method gives good results in several cases. The present formulae can be used for the cases that a rectangular cylinder sways at a center of a narrow or wide canal relative to dimension of a cylinder, at off-center location in a canal and in a restricted water with a single wall. However, the results evaluated by the present simple formulae give larger values than the numerical results of Fujino and Bai. This is caused by the assumption of the rotational motion of fluid flow in the present analysis.

From the results of numerical calculation, it is concluded that sway added mass in a restricted water is more affected by water depth than clearance between a wall and a cylinder.

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Appendix

The Dirivation of the Added Mass Formulae in the Restricted Water by Hamilton's Principle

A-1. Coordinate System and Fluid Domain

The present derivation of the sway added mass in restricted waters are essentially performed on the bases of Isshiki's method(1978).

Coordinate system is shown in Fig. 1. The sway displacement of a rectangular cylinder(It is expressed as a cylinder in the following) is $X(t)$, the sway velocity is $\dot{X}(t)$. The whole fluid domain is subdivided into 5 subdomains R_1-R_5. $\zeta_1(t)$ and $\zeta_5(t)$ are the deflections of free surface in subdomains R_1 and R_5 respectively.

R_1: $a+X\leqq x\leqq b,\ -c\leqq y\leqq \zeta_1$

R_2: $a+X\leqq x\leqq b,\ -d\leqq y\leqq -c$

R_3: $-a+X\leqq x\leqq a+X,\ -d\leqq y\leqq -c$

R_4: $-e\leqq x\leqq -a+X,\ -d\leqq y\leqq -c$

R_5: $-e\leqq x\leqq -a+X,\ -c\leqq y\leqq \zeta_5$

And S, U, S' and U' are the juncture interfaces between adjacent subdomains (Fig. 1).

A-2. Fluid Velocity in Each Subdomain

It is assumed that fluid is invicid and incompressible. The fluid velocity in each subdomain can be represented by 1 or 2 terms. If the motion of a cylinder is very small,fluid motion is mostly confined in the region near the cylinder. Thus the order of the equation representing the fluid velocity has to be higher in case of large clearance between a wall and a cylinder than in case of small clearance for which the first order equation was used by Isshiki and Murakami (1978).

(1) Fluid Velocity in R_1

Since the fluid velocity in x-direction is zero at $x=b$, u in R_1 can be described by

$u\cong u\ (x,t)=u_0(t)\ (x-b)^m$.

And since $u=\dot{X}$ at $x=a+X$, the velocity of x-direction in R_1 is given by

$$u=\frac{\dot{X}}{(a+X-b)^m}\cdot(x-b)^m \quad (1)$$

From equation (1) and 2-D continuity equation, the velocity of y-direction in R_1 is given by

$$v=-\frac{m\dot{X}}{(a+X-b)^m}\cdot(x-b)^{m-1}\ y+v_0 \quad (2)$$

where v_0 is an integral constant.

(2) Fluid Velocity in R_2

From equation(2), the fluid velocity in y-direction at $y=-c$ is given by

$$v_{y=-c}=\frac{m\dot{X}c}{(a+X-b)^m}(x-b)^{m-1}+v_0$$

If the clearance between a wall and a cylinder is small, the fluid velocity in y-direction in R_2 can be represented by a following equation.

$$v=\left[\frac{mc\dot{X}}{(a+X-b)^m}(x-b)^{m-1}+v_0\right]\frac{y+d}{d-c} \quad (3)$$

The fluid velocity in x-direction in R_2 is obtained by using the above equation,the continuity equation, and the boundary condition $u=0$ *at* $x=b$.

$$u=-\frac{1}{d-c}\left[\frac{c\dot{X}}{(a+X-b)^m}(x-b)^m+v_0\ \ (x-b)\right] \quad (4)$$

(3) Fluid Velocity in R_3

Because fluid is incompressible and $v=0$ in R_3, u is constant without regard to the position in subdomain R_3. Since the fluid velocity in x-direction at the juncture interface U is continuous, the fluid velocity is as follows,

$$u=-\frac{1}{d-c}[\dot{X}c+v_0(a+X-b)] \quad (5)$$

$$v=0. \quad (6)$$

(4) Fluid Velocity in R_5

We can describe the fluid velocity in R_5 as follows

$u=u_0(t)\ (x+e)^n$

in similar to the expression in R_1.

From the body boundary condition $u=\dot{X}$ at $x=-a+X$, u is assumed as

$$u=\frac{\dot{X}}{(e-a+X)^n}(x+e)^n \quad (7)$$

v is obtained from the continuity equation, and an integral constant is determined from the condition that the flux through S' is equal to that through U'.

$$v=-\frac{n\dot{X}}{(e-a+X)^n}(x+e)^{n-1}y+\frac{a+X-b}{e-a+X}v_0 \quad (8)$$

(5) Fluid Velocity in R_4

The fluid velocities in R_4 are obtained by the same method as in R_2

$$u=-\frac{1}{d-c}\left[\frac{c\dot{X}}{(e-a+X)^n}(x+e)^n+\frac{a+X-b}{e-a+X}v_0(x+e)\right] \quad (9)$$

$$v=\frac{d+y}{d-c}\left[\frac{nc\dot{X}}{(e-a+X)^n}\ (x+e)^{n-1}+\frac{a+X-b}{e-a+X}v_0\right] \quad (10)$$

A-3. Free Surface Condition

In this section we consider the kinematical condition on the free surface. The condition in 2-dimensional problem is

$$\frac{\partial\zeta_i}{\partial t}+u\frac{\partial\zeta_i}{\partial x}-v_i=0 \qquad \text{on } y=\zeta_i$$

Assuming that u and $\frac{\partial\zeta_i}{\partial x}$ are small, the above condition is linearized in the form,

$$\frac{\partial\zeta_i}{\partial t}=v_i.$$

By substituting (2) and (8) into this equation, we obtain

$$\frac{\partial\zeta_1}{\partial t}=-\frac{m(x-b)^{m-1}}{(a+X-b)^m}\zeta_1\dot{X}+v_0 \tag{11}$$

$$\frac{\partial\zeta_5}{\partial t}=-\frac{n(x+e)^{n-1}}{(e-a+X)^n}\zeta_5\dot{X}+\frac{a+X-b}{e-a+X}v_0 \tag{12}$$

A-4. Kinetic and Potential Energy

4.1. Kinetic energy

Kinetic energy of total system, T is described by

$$T=\frac{1}{2}M\dot{X}^2+\sum_{i=1}^{5}T_i \tag{13}$$

where M is mass per unit length of a cylinder and T_i is kinetic energy of fluid per unit length in region R_i. Kinetic energy in each region is as follows:

$$\begin{aligned}T_1&=\frac{\rho}{2}\int_{c+X}^{b}\int_{-c}^{\zeta_1}(u^2+v^2)dxdy\\&=\frac{\rho}{2}\Big[\Big\{\frac{1}{2m+1}(b-a-X)\ (\zeta_1+c)\\&\quad+\frac{m^2}{3(2m-1)}\cdot\frac{\zeta_1^3+c^3}{b-a-X}\Big\}\dot{X}^2\\&\quad+(\zeta_1^2-c^2)\dot{X}v_0+(\zeta_1+c)\ (b-a-X)v_0^2\Big]\end{aligned} \tag{14a}$$

where ρ is the density of fluid.

$$\begin{aligned}T_2&=\frac{\rho}{2}\int_{a+X}^{b}\int_{-d}^{-c}(u^2+v^2)\ dxdy\\&=\frac{\rho}{2}\Big[\Big\{\frac{c^2}{2m+1}\cdot\frac{b-a-X}{d-c}\\&\quad+\frac{m^2}{3(2m-1)}\cdot\frac{c^2(d-c)}{b-a-X}\Big\}\dot{X}^2\\&\quad-\Big\{\frac{2}{m+2}\cdot\frac{c(b-a-X)^2}{d-c}+\frac{2}{3}c(d-c)\Big\}\dot{X}v_0\\&\quad+\Big\{\frac{(b-a-X)^3}{3(d-c)}+\frac{1}{3}(d-c)(b-a-X)\Big\}v_0^2\Big]\end{aligned} \tag{14b}$$

$$\begin{aligned}T_3&=\frac{\rho}{2}\circ\frac{2b}{d-c}\Big[c^2\dot{X}^2-2c\ (b-a-X)\ \dot{X}v_0\\&\quad+(b-a-X)^2v_0^2\Big]\end{aligned} \tag{14c}$$

$$\begin{aligned}T_4&=\frac{\rho}{2}\Big[\Big\{\frac{c^2}{2n+1}\circ\frac{e-a+X}{d-c}\\&\quad+\frac{n^2}{3(2n-1)}\cdot\frac{c^2(d-c)}{e-a+X}\Big\}\dot{X}^2\\&\quad-\Big\{\frac{2c}{n+2}\cdot\frac{(e-a+X)\ (b-a-X)}{d-c}\\&\quad+\frac{2}{3}\cdot\frac{c(d-c)\ (b-a-X)}{e-a+X}\Big\}\dot{X}v_0\\&\quad+\Big\{\frac{(b-a-X)^2\ (e-a+X)}{3(d-c)}\\&\quad+\frac{(d-c)\ (b-a-X)^2}{3(e+a-X)}\Big\}v_0^2\Big]\end{aligned} \tag{14d}$$

$$\begin{aligned}T_5&=\frac{\rho}{2}\Big[\Big\{\frac{\zeta_5+c}{2n+1}\ (e-a+X)\\&\quad+\frac{n^2}{3(2n-1)}\cdot\frac{\zeta_5^3+c^3}{e-a+X}\Big\}\dot{X}^2\\&\quad-\frac{b-a-X}{e-a+X}\ (c^2-\zeta_5^2)\ \dot{X}v_0\\&\quad+\frac{(\zeta_5+c)\ (b-a-X)^2}{e-a+X}\ v_0^2\Big]\end{aligned} \tag{14e}$$

4.2. Potential energy

Potential energy of total system, V is given by the sum of potential energy in R_1 and in R_5. That is

$$V=V_1+V_5 \tag{15}$$

$$V_1=\frac{\rho g}{2}\int_{a+X}^{b}\zeta_1^2dx\approx\frac{\rho g}{2}(b-a-X)\zeta_1^2 \tag{16a}$$

$$V_5=\frac{\rho g}{2}\int_{-a}^{-a+X}\zeta_5^2dx\approx\frac{\rho g}{2}(e-a+X)\zeta_5^2 \tag{16b}$$

A-5. Application of Hamilton's Principle to the Problem of Restricted Waters

Kinetic energy and potential energy were obtained in A-4. We apply the Hamilton's principle [Hildebrand(1952)] to those energy equation.

$$L[X,\zeta_1,\zeta_5,v_0]=T-V$$

$$\begin{aligned}I&=\int_{t_1}^{t_2}\{L[X,\zeta_1,\zeta_5,v_0]+F_x\cdot X\}dt\\&=\text{stationary}\end{aligned} \tag{17}$$

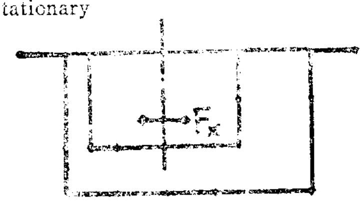

Fig. A. External Force Acting on the Body

under

$$X(t_i)=X_i,\ \zeta_1(t_i)=\zeta_{1i},\ \zeta_5(t_i)=\zeta_{5i} \text{ for } i=1,2$$

$$\frac{\partial\zeta_1}{\partial t}=-\frac{m(x-b)^{m-1}}{(a+X-b)^m}\zeta_1\dot{X}+v_0$$

$$\frac{\partial\zeta_5}{\partial t}=-\frac{n(x+e)^{n-1}}{(e-a+X)^n}\zeta_5\dot{X}+\frac{a+X-b}{e-a+X}\ v_0$$

where L is Lagrangian.

Assuming that X is small, and ζ_1, ζ_5, v_0 are also small. From these assumptions kinetic energy $T_i(i=1-5)$ are linearized as follows;

$$T_1\cong T_1^*=\frac{\rho}{2}\Big[\Big\{\frac{1}{2m+1}(b-a)c+\frac{m^2}{3(2m-1)}\cdot\frac{c^3}{b-a}\Big\}\dot{X}^2-c^2\dot{X}v_0+c(b-a)v_0^2\Big] \tag{18a}$$

$$T_2\cong T_2^*=\frac{\rho}{2}\Big[\Big\{\frac{c^2}{2m+1}\cdot\frac{b-a}{d-c}+\frac{m^2}{3(2m-1)}\cdot\frac{c^2(d-c)}{b-a}\Big\}\dot{X}^2-\Big\{\frac{2}{m+2}\cdot\frac{c(b-a)^2}{d-c}+\frac{2}{3}c(d-c)\Big\}\dot{X}v_0+\Big\{\frac{(b-a)^3}{3(d-c)}+\frac{1}{3}(d-c)(b-a)\Big\}v_0^2\Big] \tag{18b}$$

$$T_3\cong T_3^*=\frac{\rho}{2}\cdot\frac{2a}{d-c}[c^2\dot{X}^2-2c(b-a)\dot{X}v_0+(b-a)^2v_0^2] \tag{18c}$$

$$T_4\cong T_4^*=\frac{\rho}{2}\cdot\Big[\Big\{\frac{c^2}{2n+1}\cdot\frac{e-a}{d-c}+\frac{n^2}{3(2n-1)}\cdot\frac{c^2(d-c)}{e-a}\Big\}\dot{X}^2-\Big\{\frac{2c}{n+2}\cdot\frac{(e-a)(b-a)}{d-c}+\frac{2}{3}\cdot\frac{c(d-c)(b-a)}{e-a}\Big\}\dot{X}v_0+\Big\{\frac{(b-a)^2(e-a)}{3(d-c)}+\frac{(d-c)(b-a)^2}{3(e-a)}\Big\}v_0^2\Big] \tag{18d}$$

$$T_5\cong T_5^*=\frac{\rho}{2}\Big[\Big\{\frac{c}{2n+1}(e-a)+\frac{n^2}{3(2n-1)}\cdot\frac{c^3}{e-a}\Big\}\dot{X}^2-\frac{b-a}{e-a}c^2\dot{X}v_0+\frac{c(b-a)^2}{e-a}v_0^2\Big] \tag{18e}$$

Total kinetic energy T is obtained from equations (13) and (18).

$$T\cong T^*=\frac{1}{2}M\dot{X}^2+\sum_{i=1}^{5}T_i^*=\frac{1}{2}M\dot{X}^2+\frac{\rho}{2}(P\dot{X}^2-2Q\dot{X}v_0+Rv_0^2) \tag{19}$$

where

$$P=\frac{c(b-a)}{2m+1}+\frac{m^2}{3(2m-1)}\cdot\frac{c^3}{b-a}+\frac{c^2}{2m+1}\cdot\frac{b-a}{d-c}+\frac{m^2}{3(2m-1)}\cdot\frac{c^2(d-c)}{b-a}+\frac{2ac^2}{d-c}+\frac{c^2}{2n+1}\cdot\frac{e-a}{d-c}+\frac{n^2}{3(2n-1)}\cdot\frac{c^2(d-c)}{e-a}+\frac{c}{2n+1}(e-a)+\frac{n^2}{3(2n-1)}\cdot\frac{c^3}{e-a}$$

$$Q=\frac{c^2}{2}+\frac{1}{m+2}\cdot\frac{c(b-a)^2}{d-c}+\frac{1}{3}c(d-c)+\frac{2ac}{d-c}\cdot(b-a)+\frac{c}{n+2}\cdot\frac{(e-a)(b-a)}{d-c}+\frac{1}{3}\cdot\frac{c(d-c)(b-a)}{e-a}+\frac{b-a}{e-a}\cdot\frac{c^2}{2}$$

$$R=c(b-a)+\frac{(b-a)^3}{3(d-c)}+\frac{1}{3}(d-c)(b-a)+\frac{2a(b-a)^2}{d-c}+\frac{(b-a)^2(e-a)}{3(d-c)}+\frac{(d-c)(b-a)^2}{3(e-a)}+\frac{c(b-a)^2}{e-a}$$

We also linearize potential energy of eq. (16) and eq. (15).

$$V_1\cong V_1^*=\frac{\rho g}{2}(b-a)\zeta_1^2 \tag{20a}$$

$$V_5\cong V_5^*=\frac{\rho g}{2}(e-a)\zeta_5^2 \tag{20b}$$

$$V\cong V^*=V_1^*+V_5^*=\frac{\rho g}{2}[(b-a)\zeta_1^2+(e-a)\zeta_5^2] \tag{21}$$

And, by linearizing eq. (11) and (12), we obtain

$$\frac{\partial\zeta_1}{\partial t}\cong v_0 \tag{22a}$$

$$\frac{\partial\zeta_5}{\zeta t}\cong\frac{a-b}{e-a}v_0 \tag{22b}$$

$$\therefore\zeta_5=-\frac{b-a}{e-a}\zeta_1 \tag{23}$$

The linearized functional I^* is obtained by substituting the above results into eq. (17).

$$I^*[X,\zeta_1,\zeta_5,v_0]=\int_{t_1}^{t_2}[\frac{1}{2}M\dot{X}^2+\frac{\rho}{2}(P\dot{X}^2-2Q\dot{X}v_0+Rv_0^2)-\frac{\rho g}{2}(b-a)\zeta_1^2-\frac{\rho g}{2}(e-a)\zeta_5^2+F_x\cdot X]dt$$

$$=\text{stationary} \tag{24}$$

under $\begin{cases} X(t_i)=X_i,\ \zeta_1(t_i)=\zeta_{1i},\ \zeta_5(t_i)=\zeta_{5i}\ \text{for}\ i=1,2\\ \zeta_5=-\frac{b-a}{e-a}\zeta_1\\ v_0=\dot{\zeta}_1\end{cases}$

We substitute the constraints($\zeta_5=-\frac{b-a}{e-a}\zeta_1$ and $v_0=\dot{\zeta}_1$) into the functional I^* and so we may rewrite

$$I^*=\int_{t_1}^{t_2}[\frac{1}{2}(M+\rho P)\dot{X}^2-\rho Q\dot{X}\dot{\zeta}_1+\frac{\rho}{2}R\dot{\zeta}_1^2-\frac{\rho g}{2}(b-a)(1+\frac{b-a}{e-a})\zeta_1^2+F_x\cdot X\Big]dt$$

$$=\text{stationary} \tag{25}$$

under $X(t_i)=X_i,\ \zeta_1(t_i)=\zeta_{1i}$ for $i=1, 2$

The variational form of I^* can be given by the following equation.

$$\delta I^*=\int_{t_1}^{t_2}\Big[\{(M+\rho P)\ddot{X}-\rho Q\ddot{\zeta}_1-F_x\}\delta X+\{-\rho Q\ddot{X}+\rho R\ddot{\zeta}_1+\rho g(b-a)\left(1+\frac{b-a}{e-a}\right)\zeta_1\Big\}\delta\zeta_1\Big]dt=0 \tag{26}$$

From eq. (26), two coupled equations are obtained as follows:

$$(M+\rho P)\ddot{X}-\rho Q\ddot{\zeta}_1=F_x \tag{27a}$$

$$-\rho Q\ddot{X}+\rho R\ddot{\zeta}_1+\rho g(b-a)\left(1+\frac{b-a}{e-a}\right)\zeta_1=0 \tag{27b}$$

A-6. Calculation of Sway Added Mass

Let's consider the steady state of oscillation with frequency σ, then

$$F_x=\bar{F}_x e^{i\sigma t},\ X=\bar{X}e^{i\sigma t},\ \zeta_1=\bar{\zeta}_1 e^{i\sigma t}.$$

By substituting the above equations into (27a) and (27b), respectively we obtain

$$\zeta_1=\frac{QK}{RK-(b-a)\left(1+\frac{b-a}{e-a}\right)}X \tag{28}$$

$$\left\{M+\rho P+\frac{\rho Q^2 K}{(b-a)\left(1+\frac{b-a}{e-a}\right)-RK}\right\}\ddot{X}=F_x \tag{29}$$

where

$$K=\frac{\sigma^2}{g}.$$

From eq. (29) we can obtain sway added mass M_1 and sway added mass coefficient μ.

$$M_1=\rho P+\frac{\rho Q^2 K}{(b-a)\left(1+\frac{b-a}{e-a}\right)-RK} \tag{30a}$$

$$\mu=\frac{M_1}{M}=\frac{M_1}{2\rho ac}=\frac{P}{2ac}+\frac{Q^2K}{2ac\{(b-a)\left(1+\frac{b-a}{e-a}\right)-KR\}} \tag{30b}$$

As mentioned earlier, we consider the case of limiting frequencies ($\sigma\to 0$, $\sigma\to\infty$). Thus,

$$\mu_1=\frac{P}{2ac} \tag{31a}$$

$$\mu_2=\frac{P}{2ac}-\frac{Q^2}{2acR} \tag{31b}$$

where μ_1 is the sway added mass coefficient for the case $\sigma\to 0$ and μ_2 for the case $\sigma\to\infty$.