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Nonlinear Analysis of Shallow Shells and Plates by Approximate Method

Doo Sung Lee*

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Shallow 쉘과 平板에 관한 非線形 問題의 近似解析

李 斗 星

抄 錄

이 論文에서는 平板의 非線形 解析에 대한 Berger의 方程式을 쉘좌표가 直交曲線座標係로 表示되는 Shallow 쉘에 대하여 一般化하여 運動方程式을 誘導 하였다. 해석의 예로서, 이 方程式을 使用하여 固定된 境界를 가진 圓板과 Shallow 球쉘에 대한 非線形 진동문제를 해석 하였으며, 靜力學의 문제로서 圓板이 同心圓內에 均一荷重을 받을때와 中心에 集中荷重을 받을 때의 large deflection에 대하여 고찰하였고 나중問題에 대한 수치해를 구 하였다.

1. Introduction

Rigorous solutions for the nonlinear deflection of plates are difficult to obtain in general and, often, require quite laborious work even for the simplest cases. The approximate method proposed by Berger [1] for determining the deflection of plates when that deflection is of the order of magnitude of the thickness is simple, yet the results obtained by his technique have been shown by Berger to agree with the exact solutions within the accuracy required for most practical purposes when comparisons could be made.

His procedure has been generalized for various cases of plates [2, 3], and were also extended to the shallow shell problems [4]. However, so far his method is generalized to the problems of shallow spherical shells, and shallow panels, but the more general case when an orthogonal curvilinear coordinate is adopted for the shell coordinate system has not been considered.

In the present paper, it is shown that the differential equation which closely resembles Berger's original equation for the plate governs the behavior of the shell when use is made of the curvilinear coordinate system. This is established in Section 2. Using this differential equation, in Section 3, dynamic problems concerning circular plates and shallow spherical shells are discussed. In Section 4, the problem of a circular plate subjected to a uniform load over a concentric circular area is considered,

* Member. Kon-Kuk Univ.

and the final numerical example concerns the static behavior of a circular plate subjected to a point load at its center.

2. Field Equations

Let the curvilinear orthogonal coordinates (α_1, α_2) represent the principal directions of curvature at the middle surface of the shell. If u, v, and w denote the components of the displacement vector along α_1, α_2 and normal to the middle surface of the shell, respectively and, if R_1 and R_2 denote the radii of curvature corresponding to the axis α_1 and α_2 respectively, then the deformation of the middle surface pertinent to large transverse deflections is described by the equations [5]

$$\left.\begin{aligned}
\varepsilon_{11}&=\frac{1}{A_1}\frac{\partial u}{\partial\alpha_1}+\frac{1}{A_1A_1}\frac{\partial A_1}{\partial\alpha_2}v+\frac{w}{R_1}\\
&\quad+\frac{1}{2}\left(\frac{1}{A_1}\frac{\partial w}{\partial\alpha_1}\right)^2\\
\varepsilon_{22}&=\frac{1}{A_2}\frac{\partial v}{\partial\alpha_2}+\frac{1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}u+\frac{w}{R_2}\\
&\quad+\frac{1}{2}\left(\frac{1}{A_2}\frac{\partial w}{\partial\alpha_2}\right)^2\\
\varepsilon_{12}&=\left(\frac{1}{A_1}\frac{\partial v}{\partial\alpha_1}-\frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}u\right)\\
&\quad+\left(\frac{1}{A_2}\frac{\partial u}{\partial\alpha_2}-\frac{1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}v\right)\\
&\quad+\frac{1}{2}\frac{1}{A_1}\frac{\partial w}{\partial\alpha_1}\frac{1}{A_2}\frac{\partial w}{\partial\alpha_2}
\end{aligned}\right\}\qquad(2\text{-}1)$$

where ε_{11}, ε_{22} and ε_{12} are components of the inplane strain and A_1 and A_2 are Lame's parameters.

χ_{11}, χ_{22}, χ_{12}, which characterize the variation of the curvature of the middle surface induced by the deformation, are given by

$$\left.\begin{aligned}
\chi_{11}&=\frac{1}{A_1}\frac{\partial\vartheta}{\partial\alpha_1}+\frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}\phi\\
\chi_{22}&=\frac{1}{A_2}\frac{\partial\phi}{\partial\alpha_2}+\frac{1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}\vartheta\\
\chi_{12}&=\frac{1}{A_2}\frac{\partial\vartheta}{\partial\alpha_2}-\frac{1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}\phi\\
&\quad+\frac{1}{A_1}\frac{\partial\phi}{\partial\alpha_1}-\frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}\vartheta
\end{aligned}\right\}\qquad(2\text{-}2)$$

when the contributions of the "stretching" displacements u and v are neglected in case of shallow shells, ϑ and ϕ are

$$\vartheta=-\frac{1}{A_1}\frac{\partial w}{\partial\alpha_1},\quad \phi=-\frac{1}{A_2}\frac{\partial w}{\partial\alpha_2}\qquad(2\text{-}3)$$

The constitutive equations are

$$\left.\begin{aligned}
N_1&=K(\varepsilon_{11}+\nu\varepsilon_{22})\\
N_2&=K(\varepsilon_{22}+\nu\varepsilon_{11})\\
N_{12}&=K(1-\nu)\varepsilon_{12}/2
\end{aligned}\right\}\qquad(2\text{-}4)$$

where

$$K=\frac{Eh}{1-\nu^2}$$

E, h, and ν are Young's modulus, thickness of the shell, and Poisson's ratio, respectively.

The bending moments and the torque are

$$\left.\begin{aligned}
M_1&=D(\chi_{11}+\nu\chi_{22})\\
M_2&=D(\chi_{22}+\nu\chi_{11})\\
M_{12}&=D(1-\nu)\chi_{12}/2
\end{aligned}\right\}\qquad(2\text{-}5)$$

where

$$D=\frac{Eh^3}{12(1-\nu^2)}$$

Let us assume that the shell is in a state of transverse motions with an amplitude which is large compared with the amplitude of inplane motions. We may then neglect the inertia terms associated with the inplane motions.

The strain energy U_m stored in the shell in consequence of the membrane shell is

$$U_m=\frac{6D}{h^2}\int_{\alpha_1}\int_{\alpha_2}[e^2-2(1-\nu)e_2]A_1A_2d\alpha_1d\alpha_2\qquad(2\text{-}6)$$

where

$$e=\varepsilon_{11}+\varepsilon_{22},\quad e_2=\varepsilon_{11}\varepsilon_{22}-\frac{1}{4}\varepsilon_{12}^2\qquad(2\text{-}7)$$

The strain energy U_b stored in the shell in consequence of bending and torsion is

$$U_b=\frac{12}{2Eh^3}\int_{\alpha_1}\int_{\alpha_2}[M_1^2+M_2^2-2\nu M_1M_2+2(1+\nu)M_{12}^2]A_1A_2d\alpha_1d\alpha_2\qquad(2\text{-}8)$$

The change δV in the potential of external load during a virtual displacement is

$$\delta V=\int_{\alpha_1}\int_{\alpha_2}\left(q+\rho h\frac{\partial^2 w}{\partial t^2}\right)A_1A_2\delta w\,d\alpha_1d\alpha_2$$

$$+\oint_{\alpha_1}[\bar{N}_2\delta v+\bar{N}_{12}\delta u+\bar{Q}_2\delta w+\bar{M}_2\delta\psi+ \bar{M}_{12}\delta\vartheta]A_1 d\alpha_1$$
$$+\oint_{\alpha_2}[\bar{N}_1\delta u+\bar{N}_{12}\delta v+\bar{Q}_1\delta w+\bar{M}_1\delta\vartheta+ \bar{M}_{12}\delta\psi]A_2 d\alpha_2 \quad (2\text{-}9)$$

where q and ρ are the normal surface traction, and the mass density, $\bar{Q}_1$ and $\bar{Q}_2$, are the shear forces.

When the system is in equilibrium, the changes in the total potential energy of the system vanish for arbirary virtual displacement

$$\delta(U_m+U_b+V)=0$$

Following the line of reasoning of Berger, we pose $e_2=0$ in Eq. (2-6), and apply the technique of the calculus of variations to yield the following two equations of the problem

$$\frac{\partial}{\partial\alpha_1}\left[\frac{1}{A_1}\left(\frac{\partial M_1A_2}{\partial\alpha_1}+\frac{\partial M_{21}A_1}{\partial\alpha_2}+M_{12}\frac{\partial A_1}{\partial\alpha_2}-M_2\frac{\partial A_2}{\partial\alpha_1}\right)\right]+\frac{\partial}{\partial\alpha_2}\left[\frac{1}{A_2}\left(\frac{\partial M_{12}A_2}{\partial\alpha_1}+\frac{\partial M_2A_1}{\partial\alpha_2}+M_{21}\frac{\partial A_2}{\partial\alpha_1}-M_1\frac{\partial A_1}{\partial\alpha_2}\right)\right]-\frac{12D}{h^2}\left[\left(\frac{1}{R_1}+\frac{1}{R_2}\right)eA_1A_2-\frac{\partial}{\partial\alpha_1}\left(e\frac{A_2}{A_1}\frac{\partial w}{\partial\alpha_1}\right)-\frac{\partial}{\partial\alpha_2}\left(e\frac{A_1}{A_2}\frac{\partial w}{\partial\alpha_2}\right)\right]+A_1A_2\left(\rho h\frac{\partial^2 w}{\partial t^2}+q\right)=0 \quad (2\text{-}10)$$

$$\frac{\partial e}{\partial\alpha_1}=0,\quad \frac{\partial e}{\partial\alpha_2}=0 \quad (2\text{-}11)$$

If we substitute from Eqs. (2-2), (2-3), and (2-5) into Eq. (2.10), and invoke the well-known theorem that the Riemann-Christoffel tensor is zero in coordinate system in Euclidean space [6, 7], we find

$$\nabla^2\nabla^2 w-\alpha^2\left\{\nabla^2 w-\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\right\}+\frac{1}{D}\left(\rho h\frac{\partial^2 w}{\partial t^2}+q\right)=0 \quad (2\text{-}12)$$

where

$$\nabla^2=\frac{1}{A_1A_2}\left[\frac{\partial}{\partial\alpha_1}\left(\frac{A_2}{A_1}\frac{\partial}{\partial\alpha_1}\right)+\frac{\partial}{\partial\alpha_2}\left(\frac{A_1}{A_2}\frac{\partial}{\partial\alpha_2}\right)\right] \quad (2\text{-}13)$$

with

$$\alpha^2=\frac{12e}{h^2}$$

While Eqs. (2-1), (2-7) and (2-11) yield

$$\frac{1}{A_1}\frac{\partial u}{\partial\alpha_1}+\frac{1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}u+\frac{1}{A_2}\frac{\partial v}{\partial\alpha_2}+\frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}v+\frac{1}{2}\left(\frac{1}{A_1}\frac{\partial w}{\partial\alpha_1}\right)^2+\frac{1}{2}\left(\frac{1}{A_2}\frac{\partial w}{\partial\alpha_2}\right)^2+\left(\frac{1}{R_1}+\frac{1}{R_2}\right)w=\frac{\alpha^2h^2}{12} \quad (2\text{-}14)$$

If the inplane motions perpendicular to the contour of the shell vanish, Eq. (2-14) can be put into an alternative form as

$$\alpha^2=\frac{12}{h^2S}\int_{\alpha_1}\int_{\alpha_2}\left\{\frac{1}{2}\left[\left(\frac{1}{A_1}\frac{\partial w}{\partial\alpha_1}\right)^2+\left(\frac{1}{A_2}\frac{\partial w}{\partial\alpha_2}\right)^2\right]+\left(\frac{1}{R_1}+\frac{1}{R_2}\right)w\right\}A_1A_2 d\alpha_1 d\alpha_2 \quad (2\text{-}15)$$

where S denotes the total area of the shell and the integration is extended over the whole surface of the shell.

3. Method of Solution of Dynamical Problems

In this section we consider the nonlinear vibration of circular plates and shallow spherical cap. For such analysis, we apply a perturbation method of the type used by Keller and Ting [8].

For the problem involving free non-linear oscillation of a circular plate, Eq. (2-12) reduces to

$$\nabla^2\nabla^2 w-\alpha^2\nabla^2 w+\frac{\rho h}{D}\frac{\partial^2 w}{\partial t^2}=0 \quad (3\text{-}1)$$

where

$$\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$$

While Eq. (2-15) becomes

$$\alpha^2=\frac{6}{\pi h^2a^2}\int_0^{2\pi}d\theta\int_0^a\left\{\left(\frac{\partial w}{\partial r}\right)^2+\left(\frac{1}{r}\frac{\partial w}{\partial\theta}\right)^2\right\}r\,dr \quad (3\text{-}3)$$

where a is the radius of the circular plate.

We are seeking a solution to Eqs. (3-1) and (3-3) which is periodic in time with frequency ω

$$w(r,\ \theta,\ t)=w[r,\ \theta,\ t+(2\pi/\omega)] \quad (3\text{-}5)$$

To apply the perturbation method, let

$$\tau=\omega t \quad (3\text{-}6)$$

and expand w, α^2, and ω in powers of the amplitude parameter ε;

$$\left.\begin{aligned} &w(r,\ \theta,\ \tau)=\varepsilon w_1+\varepsilon^3 w_3+\varepsilon^5 w_5+\cdots \\ &\alpha^2(\tau)=\varepsilon^2 A_2+\varepsilon^4 A_4+\varepsilon^6 A_6+\cdots \\ &\omega^2(\varepsilon)=\omega_n^2(1+\varepsilon^2\Omega_2+\varepsilon^4\Omega_4+\cdots) \end{aligned}\right\} \quad (3\text{-}7)$$

where ω_n is the linear vibration frequency of the n-th normal mode.

Substituting Eqs. (3-7) into Eqs. (3-1) and (3-3), and then grouping terms of the various powers of ε yields

$$0(\varepsilon) \qquad \nabla^2\nabla^2 w_1+\frac{\rho h}{D}\omega_n^2\frac{\partial^2 w_1}{\partial\tau^2}=0 \quad (3\text{-}8a)$$

$$0(\varepsilon^2) \qquad A_2(\tau)=\frac{6}{\pi h^2 a^2}\int_0^{2\pi}d\theta\int_0^a\left\{\left(\frac{\partial w_1}{\partial r}\right)+\left(\frac{1}{r}\ \frac{\partial w_1}{\partial\theta}\right)^2\right\}rdr \quad (3\text{-}8b)$$

$$0(\varepsilon^3) \qquad \nabla^2\nabla^2 w_3+\frac{\rho h}{D}\ \omega_n^2\frac{\partial^2 w_3}{\partial\tau^2}=A_2\nabla^2 w_1-\frac{\rho h}{D}\omega_n^2\ \Omega_2\frac{\partial^2 w_1}{\partial\tau^2} \quad (3\text{-}8c)$$

$$0(\varepsilon^4) \qquad A_4(\tau)=\frac{6}{\pi h^2 a^2}\int_0^{2\pi}d\theta\int_0^a\left\{\frac{\partial w_1}{\partial r}\ \frac{\partial w_3}{\partial r}+\frac{\partial w_1}{r\partial\theta}\ \frac{\partial w_3}{r\partial\theta}\right\}rdr \quad (3\text{-}8d)$$

$$0(\varepsilon^5) \qquad \nabla^2\nabla^2 w_5+\frac{\rho h}{D}\ \omega_n^2\frac{\partial^2 w_5}{\partial\tau^2}=A_4\nabla^2 w_1+A_2\nabla^2 w_3-\frac{\rho h}{D}\ \omega_n^2\left\{\Omega_2\frac{\partial^2 w_3}{\partial\tau^2}+\Omega_4\frac{\partial^2 w_1}{\partial\tau^2}\right\} \quad (3\text{-}8e)$$

and so forth. Equations (3-8) can now be solved as a sequence of linear equations. The solution to Eq. (3-8a) is

$$w_1(r,\ \theta,\ \tau)=\varphi_n(r,\ \theta)\cos\tau \quad (3\text{-}9)$$

where

$$\varphi_n(r,\ \theta)=[AJ_p(\lambda r)+BY_p(\lambda r)+CI_p(\lambda r)+DK_p(\lambda r)]\cos p\theta$$

In the above J_p, Y_p, I_p, and K_p are Bessel functions in the usual notation and

$$\lambda^4=\frac{\rho h\omega_n^2}{D}$$

Now we suppose that the plate has no holes in the central portion and is undergoing axially symmetric oscillations while it is built-in at its rim. The boundary conditions are

$$w(r,\ \theta,\ t)=\frac{\partial w}{\partial r}[r,\theta,t]=0,\ \text{when } r=a \quad (3\text{-}10)$$

In this case φ_n depends solely on r, and is given as

$$\varphi_n(r)=J_0(\lambda r)+\beta I_0(\lambda r) \quad (3\text{-}11)$$

where

$$\beta=\frac{J_1(\lambda a)}{I_1(\lambda a)} \quad (3\text{-}12)$$

Boundary conditions (3-10) yields (see, e.g., [9]) $\lambda a=3.20$ as the first root of

$$I_0(\lambda a)J_1(\lambda a)+J_0(\lambda a)I_1(\lambda a)=0$$

corresponding to the fundamental mode of oscillations.

Having $w_1(r,\ \tau)$, $A_2(\tau)$ is determined from Eq. (3-8b)

$$A_2(\tau)=\frac{6}{a^2 h^2}(1+\cos 2\tau)\int_0^a\left(\frac{d\varphi_n}{dr}\right)^2 rdr$$

$$A_2(\tau)=\frac{6\lambda^2}{h^2}(1+\cos 2\tau)J_1^2(\lambda a) \quad (1\text{-}13)$$

Finally, $w_1(r,\ \tau)$ and $A_2(\tau)$ are substituted into the right-hand side of Eq. (3-8c) and Ω_2 is determined as follows. Since Eq. (3-8a) has non trival solutions, Eq. (3-8c) can have a solution only if the right-hand side satisfies an appropriate orthogonality condition. This condition is

$$\int_0^{2\pi}\int_0^a\left(A_2\nabla^2 w_1-\frac{\rho h}{D}\omega_n^2\ \Omega^2\frac{\partial^2 w_1}{\partial\tau^2}\right)w_1(r,\ \tau)\ rdrd\tau=0 \quad (3\text{-}14)$$

which determines Ω_2 as

$$\Omega_2=\frac{K}{\rho}\left[\frac{\lambda}{w_n}\ \frac{J_1^2(\lambda a)}{J_0(\lambda a)}\right]^2 \quad (3\text{-}15)$$

where K is as in Eq. (2-4).

With Ω_2 as given by Eq. (3-15), it is possible to solve Eq. (3-8c) and find $w_3\ (r,\ t)$, and so on. In theory this prccedure can be continued indefinitely, but in practice it is customary to stop after obtaining one or two terms in the expansion.

Carrying through similar calculations for a clamped shallow spherical shell yields results analogous to those of Eq. (3-11). If $\phi_n(r)$ denotes the modal function appropriate to the n-th mode of clamped shallow spherical shell, it is

$$\phi_n(r)=J_0(\lambda r)+\beta I_0(\lambda r)+\eta[J_1(\lambda a)+\beta I_1(\lambda a)] \tag{3-16}$$

where β satisfies the same form of equation as Eq. (3-12) and

$$\eta=\frac{12}{(h^2R^2\lambda^4-6)a\lambda} \tag{3-17}$$

In Eq. (3-17), R is the radius of the sphere, 2a is the base diameter of the shallow spherical shell, and λ is the root of the equation

$$I_0(\lambda a)J_1(\lambda a)+J_0(\lambda a)I_1(\lambda a)+2\eta J_1(\lambda a)I_1(\lambda a)=0 \tag{3-18}$$

For the value $R=\frac{a^2}{3h}$, the lowest value of the root of Eq. (3.19) gives $\frac{(\lambda a)^2}{\sqrt{12(1-\nu^2)}}=11.$ This compares well with the value 11.86 obtained using the formula by Reissner given in [14, p. 323]. The accuracy of the present solution is good for $R>\frac{a^2}{3h}$, however as R decreases beyond that the accuracy falls off.

4. Static Analysis

For the static analysis of shells, we consider the large deflection of a circular plate which is subjected to a uniform load distributed over a concentric circular area. A little more computational labor is needed for the analysis of shallow spherical shells.

We first develope a solution when a concentrated load is applied at an arbitrary point on the circular plate, which can be considered the Green's function of the problem. For this we use the singular solution for the shallow shell by Sanders [10, p. 366] (see also Simmonds and Bradley [11]). The fundamental singular solution when a concentrated load P is applied at the center of a circular plate is given as

$$-\frac{P}{\alpha^2\,2\pi\,D}[K_0(\alpha r)+\ln\alpha r] \tag{4-1}$$

This solution is transfered to the desired location on the plate so that when a concentrated load is applied at $r=s$, $\theta=0$, r in Eq. (4-1) is to be replaced by

$$\bar{r}=r^2+s^2-2rs\cos\theta \tag{4-2}$$

and then is combined with the following complementary solution to satisfy the boundary conditions

$$w_c=\sum_{m=0}^{\infty}(a_mI_m(\alpha r)+b_mK_m(\alpha r))\cos m\,\theta+\sum_{m=1}^{\infty}(c_mr^m+d_mr^{-m})\cos m\,\theta+c_0+d_0\ln r \tag{4-3}$$

When the plate is of a solid type, we have to choose

$$b_m=d_m=0\quad(m=0,1,2,\cdots)$$

Suppose that the plate is clamped at $r=a$, the boundary conditions are

$$w=\frac{\partial w}{\partial r}=0\qquad\text{at } r=a \tag{4-4}$$

To determine the unknown constants a_m and c_m in Eq. (4-3), we utilize the Graf's generalization of Newmann's addition theorem for Bessel functions [12, p. 361].

$$K_0(\alpha\bar{r})=\sum_{m=0}^{\infty}\epsilon_mK_m(\alpha r)I_m(\alpha s)\cos m\theta,\quad s>r \tag{4-5}$$

where

$$\epsilon_m=\begin{cases}1, & m=0\\ 2, & m=1,2,\cdots\end{cases} \tag{4-6}$$

The expansion of $\ln\alpha r$ is a well-known series,

$$\ln\alpha\bar{r}=\ln\alpha r-\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{s}{r}\right)^m\cos m\theta,\quad s<r \tag{4-7}$$

With the foregoing equations, we can find that the solution for a circular plate subjected to a concentrated load P at $r=s$, $\theta=0$ is given by

$$w(r,\ \theta,\ s)=\frac{P}{2\pi\alpha^2 D}\Big[\sum_{m=0}^{\infty}\frac{\epsilon_m \cos m\theta}{I_{m+1}(\alpha a)}\Big(\Big\{\frac{x^m}{\alpha a}-K_{m+1}(\alpha a)I_m(\alpha s)\Big\}I_m(\alpha r)+\Big\{I_m(\alpha s)-\xi^m I_m(\alpha a)\Big\}\frac{x^m}{\alpha a}-\frac{1}{2}\ln\Big(\frac{\xi^2+x^2-2\xi x\cos\theta}{1+\xi^2x^2-2\xi x\cos\theta}\Big)-K_0(\alpha\bar{r})\Big] \qquad (4\text{-}8)$$

where

$$x=\frac{r}{a},\ \xi=\frac{s}{a}$$

If we expand Bessel functions in Eq. (4-8) in power series, and let α approach to zero, we find that Eq. (4-8) reduces to

$$w(r,\ \theta,\ s)=\frac{Pa^2}{16\pi D}\Big[(1-x^2)(1-\xi^2)+(x^2+\xi^2-2x\xi\cos\theta)\ \ln\Big\{\frac{\xi^2+x^2-2x\xi\cos\theta}{1+\xi^2x^2-2x\xi\cos\theta}\Big\}\Big] \qquad (4\text{-}9)$$

which is the exact linear solution for the circular plate subjected to a concentrated load at an arbitrary point [13, p. 293].

The Green's function for the clamped circular plate is obtained from Eq. (4-8) by replacing θ by $\theta-\varphi$, and is the equation of the elastic surface of the plate submitted to a unit load at a fixed point $r=s$ and $\theta=\varphi$. If, therefore, some load is applied over a finite area, the corresponding deflection at any point of the plate may be obtained by integrating the product of the load and the Green's function over the loaded area.

In this way, we calculate the deflection of the plate when the uniform load q per unit area is applied over a concentric circular area of radius b. For this we need following two formulae in addition to Eqs. (4-5) and (4-7).

$$K_0(\alpha\bar{r})=\sum_{m=0}^{\infty}\varepsilon_m I_m(\alpha r)K_m(\alpha s)\cos m\theta,\quad r<s \qquad (4\text{-}10)$$

$$\ln\bar{r}=\ln r-\sum_{m=1}^{\infty}\frac{1}{m}\Big(\frac{r}{s}\Big)^m\cos m\theta,\ r<s \qquad (4\text{-}11)$$

After performing necessary integrations, we find that the normal displacement inside the loaded area is

$$w=\frac{q}{\alpha^2 D}\Big[\frac{b}{\alpha}\frac{I_0(\alpha r)}{I_1(\alpha a)}\{K_1(\alpha b)I_1(\alpha a)-K_1(\alpha a)I_1(\alpha b)\}+\frac{\alpha b^2(I_0(\alpha r)-I_0(\alpha a))+2bI_1(\alpha b)}{2\alpha^2 a I_1(\alpha a)}-\frac{b^2}{2}\ln\frac{b}{a}+\frac{b^2-r^2}{4}-\frac{1}{\alpha^2}\Big]\ r<b \qquad (4\text{-}12)$$

Outside the loaded region, we have

$$w=\frac{q}{\alpha^2 D}\Big[-\frac{b}{\alpha}\frac{I_1(\alpha b)}{I_1(\alpha a)}\{K_0(\alpha r)I_1(\alpha a)+K_1(\alpha a)I_0(\alpha r)\}+\frac{\alpha b^2(I_0(\alpha r)-I_0(\alpha a))+2bI_1(\alpha b)}{2\alpha^2 a I_1(\alpha a)}-\frac{b^2}{2}\ln\frac{r}{a}\Big],\ b<r \qquad (4\text{-}13)$$

The deflection of the plate when a concentrated load is applied at the center can be obtained from either Eq. (4-13) or Eq. (4-8) and is equal to

$$w=\frac{P}{2\pi\alpha^2 D}\Big[-\frac{K_0(\alpha r)I_1(\alpha a)+K_1(\alpha a)I_0(\alpha r)}{I_1(\alpha a)}+\frac{I_0(\alpha r)-I_0(\alpha a)+1}{\alpha a I_1(\alpha a)}-\ln\frac{r}{a}\Big] \qquad (4\text{-}14)$$

The second Berger equation to Eq. (4-14) obtained using Eq. (3-3) is

$$\Big(\frac{Pa^2}{Dh\pi}\Big)^2=\frac{\frac{1}{3}(\alpha a)^6}{\Big[\frac{2-I_0(\alpha a)}{\alpha a\, I_1(\alpha a)}-\frac{K_1(\alpha a)}{I_1(\alpha a)}-\frac{(I_0(\alpha a)-1)^2}{2I_1^2(\alpha a)}+\ln\frac{\alpha a}{2}+\gamma\Big]} \qquad (4\text{-}15)$$

where γ denotes Euler's constant (0.5772157). From this equation one finds the proper value of αa to be used in Eq. (4-14) for the given

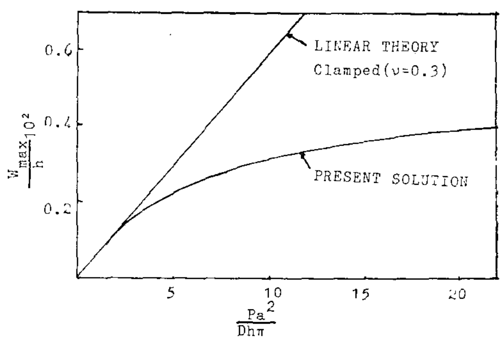

Fig. 1.

load. The resulting solution for the deflection at the center of the plate is plotted in Fig. 1.

The second Berger equation to Eqs. (4-12) and (4-13) for determing the coupling parameter α is

$$\left(\frac{qa^2}{Dh}\right)^2=\frac{(a\alpha)^6}{12}\Big[\frac{b^2}{I_1(\alpha a)}\Big\{K_1(\alpha a)I_1(\alpha b)-I_1(\alpha a)K_1(\alpha b)-\frac{b}{2a}\Big\}\Big\{\frac{b}{\alpha}I_0(\alpha b)-\frac{3}{\alpha^2}I_1(\alpha b)\Big\}-\frac{a^2b^2}{4}\Big\{\frac{2I_1(\alpha b)-\alpha bI_0(\alpha a)}{\alpha aI_1(\alpha a)}-\frac{b}{\alpha a^2}\Big\}^2+\frac{3}{16}b^4+\frac{b^4}{8\alpha^2a^2}-\frac{b^2}{2\alpha^2}-\frac{b^4}{4}\ln\frac{b}{a}\Big] \tag{4-16}$$

Notice that if we keep πqb^2 constant while letting b approach zero in the above equation, Eq. (4-16) reduces to Eq. (4-15). Also if b is replaced by a in Eq. (4-16), it becomes the equation for the uniformly loaded circular plate and completely agrees with Eq. (22) of [1].

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