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Necessary Conditions for Optimal Actuator and Sensor Locations for Suboptimal Control of Distributed Parameter Systems

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연속계 제어시스템의 설계를 위한 구동장치와 측정장치의 최적위치 선정

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초 록

연속계의 동적 시스템을 제어하고저 할 때에는 구동장치와 측정장치를 어느 위치에 두느냐에 따라 시스템의 제어 성능이 상당히 달라지기 때문에 위치선정 문제가 매우 중요한 설계변수의 하나이다.

본 연구에서는 일반 연속계의 제어시스템을 설계하고저 할 때를 고려하여, 그들의 최적 위치를 결정할 수 있는 조건식을 최적제어이론을 바탕으로 유도하였다.

1. Introduction

In most physical systems the parameters which determine the behavior of the process have spatial and temporal distribution. One particularly important consideration in controlling such system is to achieve better control effectiveness with fewer actuators and sensors needed to implement a control law. A rational approach to achieve this end is to optimize the locations of a prescribed number of the actuators and sensors. This optimal actuator and sensor location problem has been treated by various authors from different viewpoints. (1)-(5) In previous studies the optimal actuator and sensor location problems have been treated separately, and furthermore, the controller scheme for feedback implementation has not been specifically included in their design considerations.

This paper considers an optimal output feedback control of distributed parameter system and presents a design methodology for determining the optimal controller gain as well as the optimal actuator and sensor locations. Necessary conditions for determining such optimal variables are derived from minimization of a prescribed quadratic criterion on the state and control input. The derivation is based upon the method of Levine and Athans (6) who have supplied necessary conditions for optimality of the output feedback gains for a linear time-invariant control system.

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2. Control System Formulation

Consider the class of distributed parameter system thatcan be described by

$$L[v(\phi,t)]+\frac{\partial v(\phi,t)}{\partial t}=\sum_{i=1}^{N}\delta(\phi-\phi_i^a)f_i(t) \qquad (1)$$

over the domain $0<\phi<1$ and $t\geq 0$. Typical example of the distributed parameter system governed by Eq. (1) is one dimensional heat conduction problem whose practical applications can be found for the heating of steel ingots in soaking pit furnaces in steelmaking process.

In the above $v(\phi,t)$ and its derivatiives are assumed to be square integrable over the domain for every fixed time $t\geq 0$. The operator L is a linear differential operator over the spatial variable ϕ and spatially varying coefficients are allowed in the operator. The operator consists of derivative through order 2r with respect to the spatial coordinates. The right hand side of (1) represents the pointly distributed control inputs provided by N point actuators which are located at points (ϕ_i^a ; $i=1,2,\cdots N$). Output measurements are taken by P point sensors located at points(ϕ_j^s ; $j=1,2,\cdots p$) which measure

$$y_j(t)=v(\phi_j^s,t) \quad j=1,2,\cdots p \qquad (2)$$

At each point of the boundary there are r boundary conditions of the type

$$B_i[v(\phi,t)]=0 \quad i=1,2,\cdots r \qquad (3)$$

where B_i are linear homogeneous differential operators containing derivatives normal to the boundary and along the boundary of order through 2r-1.

The solution of the problem (1) and (3) described above may be exact or approximate, depending upon whether or not the function satisfies the differential equation (1) or boundary conditions (3). In case of seeking an approximate solution a finite dimensional model may be formed by choosing a set of linearly independent functions $v_m(\phi)$, $m=1,2,\cdots l$,

$$v(\phi,t)=\sum_{m=1}^{l}v_m(\phi)q_m(t) \quad \phi\varepsilon[0,1], \quad t\geq 0 \qquad (4)$$

where $q_m(t)$ are time-dependent generalized coordinates. For simplicity of analysis, if these approximating functions can be chosen to be the first l eigenfunctions which satisfy the homogenious equation of (1) and the boundary condition (3), then a standard technique leads to the following time-dependent modal equations

$$\dot{q}_m(t)+\lambda_m q_m(t)=\sum_{i=1}^{N}v_m(\phi_i^a)f_i(t),$$
$$m=1,2,\cdots l \qquad (5)$$

where $\lambda_m, =\langle v_m, L[v_m]\rangle$ is the eigenvalue of the corresponding eigenfunction $v_m(\phi)$. Defining the state variables $x(t)=\{q_1(t),q_2(t), \ldots q_l(t)\}^T$, the modal equations (5) and the measurement equations in (2) can be readily put into the following state equations :

$$\dot{\underset{\sim}{x}}(t)=A\underset{\sim}{x}(t)+B(\underset{\sim}{\phi}^a)\underset{\sim}{u}(t) \qquad (6)$$
$$\underset{\sim}{y}(t)=C(\phi^s)\underset{\sim}{x}(t)$$

The control input matrix $B(\phi^a)$ is a $L\times N$ matrix whose rows are the mode shape functions $v_m(\phi^a)$ evaluated at the actuator locations, while the sensor output matrix $C(\phi^s)$ is a $p\times L$ matrix whose columms are the mode shape function $v_m(\phi^s)$ evaluated at the sensor locations.

3. Suboptimal Controller

A design criterion to be used for the control system design is to choose the control $u(t)$ which minimizes a quadratic performance measure of the form.

$$J=\frac{1}{2}\int_0^\infty[\underset{\sim}{x}^T(t)Q\underset{\sim}{x}(t)+\underset{\sim}{u}^T(t)R\underset{\sim}{u}(t)]dt \qquad (7)$$

where the matrices Q and R are positive semidefinite and positive definite, respectively. The matrix Q permits the state variables to be weighted relative to each other, while R is included to constrain the control input, $\underset{\sim}{u}(t)$. For a completely controllable system the suboptimal controller scheme may be expressed by (6)

$$\underset{\sim}{u}(t)=F\underset{\sim}{y}(t)=FC(\underset{\sim}{\phi}^s)\underset{\sim}{x}(t) \qquad (8)$$

where the $N\times P$ matrix F is an output feed back gain matrix. If the control law (8) is substituted, the solution of the resulting system equation can be written as

$$\underset{\sim}{x}(t)=\exp([A+BFC]t)\underset{\sim}{x}(0) \qquad (9)$$

Examination of (7) and (9) shows the dependency of the performance measure on the initial state $\underset{\sim}{x}(0)$. Following Levine and Athans (6), this dependency may be eliminated and the following modified performance criteria may be used :

$$\bar{J}=E\{J\}=\frac{1}{2}t_r\{K(F,B,C)\}$$

where $K(F,B,C\}=\frac{1}{2}\int_0^\infty\{\exp([A+BFC]t)^T$

$$(Q+C^TF^TRFC)\exp([A+BFC]t)\}dt \qquad (10)$$

In the above the operator $E\{\cdot\}$ indicates taking the expected value over all possible $\underset{\sim}{x}(0)$and $t_r\{\cdot\}$the trace of a square matrix.

4. Derivation of Necessary Conditions

The performance measure $\bar{J}$ in (10) is a real function of $N\times P$ variables, the f_{ij}, N variables ϕ_i^a and P variables ϕ_i^s. Necessary conditions for F, $\underset{\sim}{\phi}^a$ and $\underset{\sim}{\phi}^s$ to minimize this function are that

$$\frac{\partial\bar{J}}{\partial F}\bigg|_{F*}=0 \qquad (11)$$

$$\frac{\partial\bar{J}}{\partial\underset{\sim}{\phi}^a}\bigg|_{\underset{\sim}{\phi}^{*a}}=\left[\frac{\partial\bar{J}}{\partial B}\right]^T\left[\frac{\partial B}{\partial\underset{\sim}{\phi}^a}\right]\bigg|_{\underset{\sim}{\phi}^{*a}}=0 \qquad (12)$$

$$\frac{\partial\bar{J}}{\partial\underset{\sim}{\phi}^s}\bigg|_{\underset{\sim}{\phi}^{*s}}=\left[\frac{\partial\bar{J}}{\partial C}\right]\left[\frac{\partial C}{\partial\underset{\sim}{\phi}^s}\right]^T\bigg|_{\underset{\sim}{\phi}^{*s}}=0 \qquad (13)$$

Levine and Athans (6) derived a necessary condition for F^* to minimize the $\bar{J}$ in (10), using the first condition (11), and later, Hutcheson (7) also obtained the same result by a more simple matrix algebra. This condition is given by

$$(RFC+B^TK)LC^T=0 \qquad (14)$$

In the above K is a positive semidefinite solution of

$$K[A+BFC]+[A+BFC]^TK+Q+C^TF^TRFC=0 \qquad (15)$$

and L is a positive definite solution of

$$L[A+BFC]^T+[A+BFC]L+I=0 \qquad (16)$$

Following Hulcheson's method (7), the necessary conditions for $\underset{\sim}{\phi}^{*a}$ and $\underset{\sim}{\phi}^{*s}$ to minimize $\bar{J}$ can be easily obtained. The second necessary condition can be derived using the conditions (12), (15) and (16). Let β be a scalar parameter in B. Then differentiating (15) with respect to β gives

$$\frac{\partial K}{\partial\beta}[A+BFC]+K\frac{\partial B}{\partial\beta}FC+\left[\frac{\partial B}{\partial\beta}FC\right]^TK+[A+BFC]^T\frac{\partial K}{\partial\beta}=0 \qquad (17)$$

Postmultiplying (17) by L and taking the traces

$$t_r\left\{\frac{\partial K}{\partial\beta}[A+BFC]L\right\}=-t_r\left\{K\frac{\partial B}{\partial\beta}FCL\right\} \qquad (18)$$

Premultiplying (16) by $\frac{\partial K}{\partial\beta}$ and taking the traces

$$t_r\left\{\frac{\partial K}{\partial\beta}[A+BFC]L\right\}=-\frac{1}{2}t_r\left\{\frac{\partial K}{\partial\beta}\right\} \qquad (19)$$

Combining (18) and (19) yields the following result

$$\frac{1}{2}t_r\left\{\frac{\partial K}{\partial\beta}\right\}=t_r\left\{K\frac{\partial B}{\partial\beta}FCL\right\} \qquad (20)$$

Since $\frac{\partial\bar{J}}{\partial\beta}=\frac{1}{2}t_r\left\{\frac{\partial K}{\partial\beta}\right\}=t_r\left\{\frac{\partial\bar{J}}{\partial B}\frac{\partial B^T}{\partial\beta}\right\}$, the following relationship holds from (20) :

$$t_r\left\{\frac{\partial\bar{J}}{\partial B}\frac{\partial B^T}{\partial\beta}\right\}=t_r\left\{K\frac{\partial B}{\partial\beta}FCL\right\}$$

$$=t_r\left\{LC^TF^T\frac{\partial B^T}{\partial\beta}K\right\} \qquad (21)$$

Using the trace property $\partial t_r\{XY^TZ\}/\partial Y=XZ$, and differentiating (21) with respect to the matrix $\frac{\partial B}{\partial\beta}$ lead to

$$\frac{\partial \bar{J}}{\partial B}=KLC^TF^T \tag{22}$$

Combining (12) and (22) yields one of the optimality conditions.

$$\left[KLC^TF\right]_i^T \frac{\partial B_i}{\partial \phi_i^a}=0 \quad i=1,2\ldots N \tag{23}$$

where$[\cdot]_i$ of the left hand side of the equation indicates the ith row of the matrix and B_i is the ith column of the $B(\underset{\sim}{\phi}^a)$ matrix

In the exactly same way, the other optimality condition can be derived using (13), (15) and (16). Here only the result is presented, omitting the derivation.

$$[F^T(RFC+B^TK)L]_j \frac{\partial C_j^T}{\partial \phi_j^S}=0 \quad j=1,2\ldots p \tag{24}$$

where $[\cdot]_j$ of the left hand side of the equation indicates the jth row of the matrix, while C_j is the jth row of the $C(\underset{\sim}{\phi}^S)$ matix. In summary, the necessary conditions for $F, \underset{\sim}{\phi}^a$ and $\underset{\sim}{\phi}^S$ to be optimal can be written as

$$(RFC+B^TK)LC^T=0$$

$$\left[(KLC^TF^T)\right]_i^T \frac{\partial B_i}{\partial \phi_i^a}=0, \quad i=1,2,\ldots N$$

$$[F^T(RFC+B^TK)L]_j \frac{\partial C_j^T}{\partial \phi_j^S}=0, \quad j=1,2,\ldots p$$

where K and L must satisfy the following algebraic matrix equations:

$$K[A+BFC]+[A+BFC]^TK+Q$$
$$+C^TF^TRFC=0$$

$$L[A+BFC]^T+[A+BFC]L+I=0$$

It is assumed here that $F, \underset{\sim}{\phi}^a$ and $\underset{\sim}{\phi}^s$ exist, which stabilize $[A+BFC]$. If no such F $\underset{\sim}{\phi}^a$ and $\underset{\sim}{\phi}^S$ exist, then $\bar{J}$ is infinite and this optimization problem is meaningless. Note that the above conditions are only necessary conditions and thus there may exist several other nonoptimal solutions which satisfy these conditions.

Further study should be focused on developing an efficient computer program for solving these nonlinear algebraic matrix equation. The computing endeavour depends not only upon efficiency of the scheme itself but also upon how many approximating functions are to be included in the control system formulation. In a work following this study, numerical simulation study will be illustrated to determine optimal actuator and sensor locations for one dimensional heat conduction problem.

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